

Meta-Cayley Graphs on Dihedral Groups

by
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The logo of the University of the Western Cape, featuring a stylized building facade with columns and a pediment, rendered in a light blue color. Below the graphic, the text "UNIVERSITY of the WESTERN CAPE" is displayed in a serif font, with "of the" in italics.

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Abstract

The pursuit of graphs which are vertex-transitive and non-Cayley on groups has been ongoing for some time. There has long been evidence to suggest that such graphs are a very rarity in occurrence. Much success has been had in this regard with various approaches being used. The aim of this thesis is to find such a class of graphs. We will take an algebraic approach. We will define Cayley graphs on loops, these loops necessarily not being groups. Specifically, we will define meta-Cayley graphs, which are vertex-transitive by construction. The loops in question are defined as the semi-direct product of groups, one of the groups being \mathbb{Z}_2 consistently, the other being in the class of dihedral groups. In order to prove non-Cayleyness on groups, we will need to fully determine the automorphism groups of these graphs. Determining the automorphism groups is at the crux of the matter. Once these groups are determined, we may then apply Sabidussi's theorem. The theorem states that a graph is Cayley on groups if and only if its automorphism group contains a subgroup which acts regularly on its vertex set.

Declaration

I hereby declare that this thesis is my own work, that it has not been submitted for any degree or examination at any other academic institution, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.



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January 2017

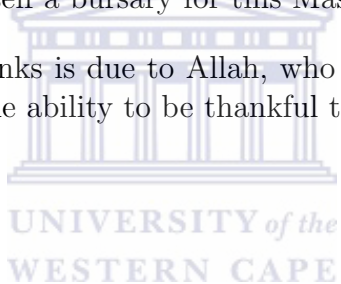
Signed:

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Chapter 1

Introduction

Vertex-transitivity in graphs represents a measure of symmetry, which is defined by the graph having an automorphism group which acts transitively on the vertex set. Archetypal vertex-transitive graphs are those defined as Cayley graphs on groups.

While it can be proved, without difficulty, that all Cayley graphs on groups are vertex-transitive, the reverse implication does not hold. A counter example for the reverse implication is the well-known Petersen graph.

Through computational observations, it has been conjectured by Ivanov and Praeger [7] that

$$\lim_{n \rightarrow \infty} \frac{\text{cay}(n)}{\text{vtr}(n)} = 1,$$

where $\text{vtr}(n)$ and $\text{cay}(n)$ denote the number of isomorphism types of vertex-transitive and Cayley graphs on groups, respectively, with order at most $n \geq 1$. In other words, it is conjectured that the majority of vertex-transitive graphs are Cayley on groups. That is, the majority of vertex-transitive graphs can be represented by a group and a Cayley set. Much earlier, in [9] Marušič asked for which positive integers n does there exist a vertex-transitive graph on n vertices which is not Cayley on groups. It has been noted that there are indications that vertex-transitive graphs that are not Cayley on groups are a rarity in occurrence. Further, finding classes of graphs which are vertex-transitive and non-Cayley on groups has proved to be no simple matter. The determination of vertex-transitive graphs that are not Cayley on groups has thus raised a lot of attention. Notable success has been had in the construction

of such graphs; see, for instance, [2, 6, 8, 10, 11, 18, 20]. Much of the success had in this line has been in part due to Sabidussi's theorem in [19]. The theorem states that a graph is Cayley on a group if and only if the automorphism group contains a subgroup which acts regularly on the vertex set.

The method of defining a Cayley graph on a group has been generalised to defining a Cayley graph on a groupoid in [13]. In fact, it was shown that every graph is representable on some groupoid. As mentioned, every Cayley graph on a group is vertex-transitive. However, to ensure vertex-transitivity of a Cayley graph on a groupoid, all we require is that the groupoid is a loop, and the Cayley set conforms to a property called quasi-associativity, a concept introduced by Gauyacq in [5].

In [16], meta-Cayley graphs were introduced. These graphs are vertex-transitive by construction. The construction involves the semi-direct product of two groups of which the twisting map satisfies a certain weak condition, such that the semi-direct product is a loop. Also, the Cayley set on the semi-direct product takes a particular form to ensure quasi-associativity. Of course this semi-direct product need not be a group. Thus, an avenue for the determination of vertex-transitive graphs which are non-Cayley on groups was introduced.

The purpose of this thesis is to define a class of meta-Cayley vertex-transitive graphs and then prove these graphs to be non-Cayley on groups. In Chapter 2, we present the necessary preliminaries relating to graph theory and groupoids. In Chapter 3 we explain how graphs are constructed on groupoids. Also, we present meta-Cayley graphs. In Chapter 4, we construct our vertex-transitive meta-Cayley graphs. In Chapter 5, we fully determine the automorphism groups of our constructed graphs and prove that they are non-Cayley on groups by applying Sabidussi's theorem. Henceforth, we will refer to vertex-transitive graphs which are non-Cayley on groups by the acronym VTNCGs introduced by Watkins in [20].

Chapter 2

Preliminaries

In this chapter, we present the necessary preliminaries relating to graph and group theory. Naturally we begin with the formal definition of a graph. Thereafter, we define various concepts. At the crux of what we aim to achieve in this thesis is the determination of automorphism groups of graphs. Therefore, we define automorphisms on graphs which is also necessary in introducing the concept of vertex-transitivity. Also included are the definitions of certain algebraic structures, particularly groupoids and loops, as well as the semi-direct product of groups. We conclude the chapter with the definition of a class of graphs which we will consider in this thesis, that is, the generalised Petersen graphs.

2.1 Basic graph theory

Formally, graphs are defined as follows.

Definition 2.1. Let V be a non-empty set and E a relation on V . We call the pair $\Gamma = (V, E)$ a *digraph*. If E is irreflexive and symmetric, we call the pair $\Gamma = (V, E)$ a *graph*.

In a graph $\Gamma = (V, E)$ the elements of V are called *vertices* and the elements of E are called *edges*. Edges are denoted as $[x, y]$ where $x, y \in V$. We say that x and y are *adjacent*, x and y are *incident*, or that x and y are *neighbours*. Similarly, edges

can also be thought of as incident or adjacent if they share a common vertex. The *degree* or *valency* of a vertex is the number of edges incident with it. If the degree is the same for each vertex in the graph, then we refer to the degree of the graph.

We now define various types of sequences of vertices.

Definition 2.2. Let $\Gamma = (V, E)$ be a graph.

- (a) A *walk* is a sequence of vertices v_0, v_1, \dots, v_k such that $[v_i, v_{i+1}] \in E$ for every $i \in \{0, 1, \dots, k-1\}$. k is called the *length* of the walk.
- (b) A *trail* is a walk in which every edge is distinct.
- (c) A *path* is a trail in which every vertex in $\{v_0, v_1, \dots, v_{k-1}\}$ is distinct.
- (d) A *cycle* is a path in which $v_k = v_0$. An n -*cycle* is a cycle with n vertices.

Let $\Gamma = (V, E)$ be a graph. Define a relation \sim on V by $x \sim y$ if and only if there exists a path from x to y . If $x \sim y$ then x and y are said to be *connected*. It is not hard to see that \sim is an equivalence relation. Each equivalence class $[x]$ is called a *component* of the graph. If a graph has more than one component then it is *disconnected*, otherwise it is *connected*.

Graph homomorphisms can naturally be defined as mappings which preserve the relation on the vertex set. That is, graph homomorphisms are edge-preserving.

Definition 2.3. Let Γ and Λ be graphs. Let ϕ be a mapping, $\phi : V(\Gamma) \longrightarrow V(\Lambda)$.

- (a) If $[x, y] \in \Gamma$ implies $[\phi(x), \phi(y)] \in \Lambda$ then ϕ is called a *homomorphism*. Γ and Λ are said to be *homomorphic*.
- (b) If ϕ is a bijective homomorphism and ϕ^{-1} is a homomorphism then ϕ is called an *isomorphism*. Γ and Λ are said to be *isomorphic*, denoted as $\Gamma \cong \Lambda$.
- (c) If ϕ is an isomorphism from a graph to itself then ϕ is called an *automorphism*.

Remark 2.4. The automorphisms of a graph Γ form a group under composition. The automorphism group is denoted as $\text{Aut } \Gamma$.

In this thesis, we will be proving isomorphic relations between graphs. We present the following lemma which will prove useful in this regard.

Lemma 2.5. *Let Γ and Λ be graphs with finite vertex sets and let $\phi : V(\Gamma) \rightarrow V(\Lambda)$ be a bijective homomorphism. If $|E(\Gamma)| = |E(\Lambda)|$ then $\Gamma \cong \Lambda$.*

Proof. Let $E(\Gamma) = \{e_1, e_2, \dots, e_k\}$. Then $\phi(E(\Gamma)) = \{\phi(e_1), \phi(e_2), \dots, \phi(e_k)\}$. For each i let $e_i = [x_i, y_i]$ so that $\phi(e_i) = [\phi(x_i), \phi(y_i)]$. Since ϕ is a bijection on the vertex sets, each $\phi(e_i)$ is distinct. Therefore, $|E(\Lambda)| = |E(\Gamma)| = |\phi(E(\Gamma))|$ so that ϕ is onto with respect to edges. By the pigeonhole principle, ϕ^{-1} is a homomorphism so that ϕ is an isomorphism. \square

Remark 2.6. For automorphisms, the order of the edge sets are obviously the same. Thus we will not explicitly apply Lemma 2.5 when proving that certain maps are automorphisms.

Since the automorphisms form a group, we consider the action of this group on the set of vertices or edges. For any vertex $x \in V$ and automorphism $\gamma \in \text{Aut } \Gamma$ we define the action of γ on x as $\gamma(x)$. Similarly for an edge $[x, y] \in E$ we define the action of γ on $[x, y]$ as $\gamma([x, y]) = [\gamma(x), \gamma(y)]$. If for any two vertices $x, y \in V$ there exists an automorphism $\gamma \in \text{Aut } \Gamma$ such that $\gamma(x) = y$, then the group $\text{Aut } \Gamma$ is said to act transitively on V . If for any two edges $[x_1, y_1], [x_2, y_2] \in E$ there exists an automorphism $\gamma \in \text{Aut } \Gamma$ such that $\gamma([x_1, y_1]) = [x_2, y_2]$, then the group $\text{Aut } \Gamma$ is said to act transitively on E . This leads us to the definition of vertex-transitivity and edge-transitivity.

Definition 2.7. Let $\Gamma = (V, E)$ be a graph.

- (a) If $\text{Aut } \Gamma$ acts transitively on $V(\Gamma)$ then Γ is said to be *vertex-transitive*.
- (b) If $\text{Aut } \Gamma$ acts transitively on $E(\Gamma)$ then Γ is said to be *edge-transitive*.
- (c) Let H be a subgroup of $\text{Aut } \Gamma$ which acts transitively on $V(\Gamma)$. If H has the same order as $V(\Gamma)$ then H is said to act *regularly* on $V(\Gamma)$.

We immediately note that if a graph is vertex-transitive then every vertex has the same degree. Also, every vertex is contained in the same amount of cycles of each size. For these reasons, we think of vertex-transitivity as a measure of symmetry within graphs.

Let $\Gamma = (V, E)$ be a graph. Define a relation \sim on V by $x \sim y$ if and only if there exists an automorphism $\gamma \in \text{Aut } \Gamma$ such that $\gamma(x) = y$. It is easy to see that \sim is an

equivalence relation. The equivalence class containing a vertex x is called the *orbit* of x . If Γ is vertex-transitive then the orbit of any $x \in V$ is V itself.

2.2 Basic algebraic definitions

As we know, a group is a set paired with a binary operation which satisfies certain conditions. That is, a group contains an identity element, is closed under inverses, has full element-wise associativity and the binary operation is closed. Not every set paired with a binary operation satisfies these conditions. We present some definitions of structures which satisfy less than the conditions required for a group, beginning with the simplest version of a set paired with a closed binary operation.

Definition 2.8. Let A be a set and \oplus be a closed binary operation on A . The pair (A, \oplus) is called a *groupoid*.

Note that groupoids are often written as just A instead of (A, \oplus) .

For our purposes, we will require some structure. Loops, defined below, will suffice.

Definition 2.9. Let A be a groupoid. If for every $a, b \in A$ there exists a unique x satisfying $ax = b$ then we say that A is a *left quasi-group*. If for every $a, b \in A$ there exists a unique y satisfying $ya = b$ then we say that A is a *right quasi-group*. If A is both a left quasi-group and a right quasi-group then A is called a *quasi-group*.

Definition 2.10. Let A be a [left][right] quasi-group. If A has an identity element then A is called a [left][right] *loop*.

Every element in a loop has a unique *right inverse* and a unique *left inverse*. Due to the lack of associativity, these inverses are not necessarily the same. Let A be a loop with $U \subseteq A$, $a \in A$ and e is the identity element. By a^{-1} we mean that $aa^{-1} = e$. By U^{-1} we mean the set $\{u^{-1} \in A : uu^{-1} = e, u \in U\}$.

It is possible to combine two groups to form a product of groups. Naturally, for two groups G and H , this is done by taking $G \times H$ as the set of elements and defining the operation as $(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2)$. This is known as the *direct product* of two groups, which itself is a group. We may also take what is called the semi-direct

product of two groups. This is similar to the direct product, except that it involves a mapping from G to $\text{Aut } H$.

Definition 2.11. Let G and H be groups. Let $f : G \rightarrow \text{Aut } H$ be a mapping where $f(x)$ is denoted as f_x . The set $G \times H$ together with the binary operation defined by $(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1f_{g_1}(h_2))$ is called a *semi-direct product of G and H* . The semi-direct product is denoted as $G \times_f H$.

Remark 2.12. The semi-direct product is not necessarily a group. However, it is easily shown that $G \times_f H$ forms a group if f is a group homomorphism. The mapping f is often referred to as a “twisting” map.

A class of graphs which will be considered in this thesis, is the generalised Petersen graphs. As made clear by the name, these graphs are generalisations of the well-known Petersen graph. They are defined as follows.

Definition 2.13. Let $G(n, r)$ be a graph with vertex set

$$V = \{v_i, u_i : i \in \mathbb{Z}_n\}$$

and edge set

$$E = \{[v_i, v_{i+1}], [u_i, u_{i+r}], [v_i, u_i] : i \in \mathbb{Z}_n\},$$

where all subscripts are modulo n , $r < n$ and $r \neq \frac{n}{2}$ if n is even. $G(n, r)$ is called a *generalised Petersen graph*. The set of edges $\{[v_i, v_{i+1}] : i \in \mathbb{Z}_n\}$ is called the *outer rim*, $\{[u_i, u_{i+r}] : i \in \mathbb{Z}_n\}$ is called the *inner rim*, and edges in $\{[v_i, u_i] : i \in \mathbb{Z}_n\}$ are called *spokes*.

Example 2.14. The graph presented in the figure below is the generalised Petersen graph $G(8, 3)$. The outer rim, inner rim and spokes are easily identifiable.

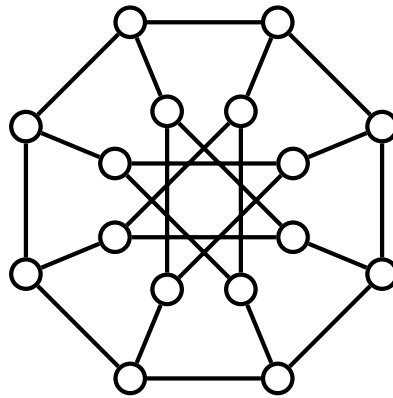


Figure 2.1: The generalised Petersen graph $G(8,3)$.



Chapter 3

Graphs on groupoids

In this Chapter, we present how graphs are defined on algebraic structures, most notably, groupoids. Furthermore, we present meta-Cayley graphs. The class of graphs which we will introduce in the next chapter will be of this type.

3.1 Cayley graphs

Graphs can be defined on various algebraic structures. Most commonly, graphs are defined on groups. In doing so, the vertices of the graph are naturally defined as the elements of the group. In order to define the irreflexive and symmetric relation, so that Definition 2.1 is satisfied, subsets of the group elements are chosen which satisfy certain conditions. These subsets are known as Cayley sets and are defined as follows.

Definition 3.1. Let G be a group and let 1_G be the identity element of G . Let $S \subset G$. If

- (i) $1_G \notin S$ and
- (ii) $s \in S$ implies $s^{-1} \in S$,

then S is called a *Cayley set on G* .

Now let G be a group and S a Cayley set on G . We define a relation E on G by $(x, y) \in E$ if and only if $y = xs$ where $s \in S$. We then have $ys^{-1} = x$ so that $(y, x) \in E$ since $s^{-1} \in S$. Therefore E is symmetric. Also, since $1_G \notin S$ it is impossible that $x = xs$ for some $s \in S$. Therefore $(x, x) \notin E$ and E is irreflexive. This brings us to the following definition of a Cayley graph.

Definition 3.2. Let G be a group and S a Cayley set on G . The graph Γ consisting of the vertex set

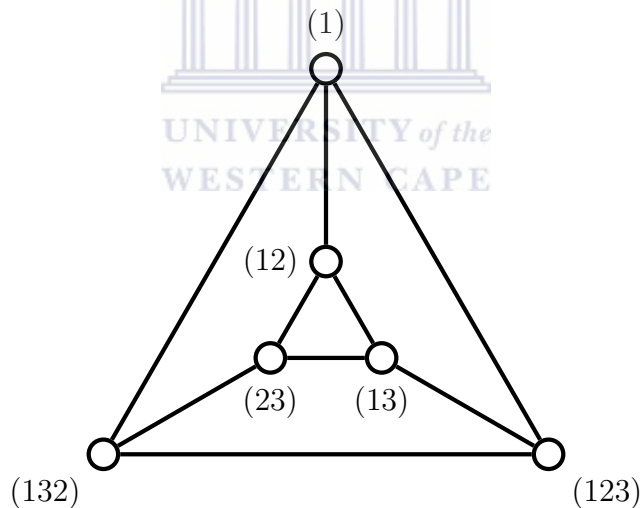
$$V(\Gamma) = G$$

and edge set

$$E(\Gamma) = \{[x, xs] : x \in G, s \in S\}$$

is called a *Cayley graph on G* and is denoted as $\text{Cay}(G, S)$.

Example 3.3. Consider the symmetric group S_3 . $U = \{(12), (123), (132)\}$ is a Cayley set on S_3 since it is closed under inverses and does not contain the identity element. The Cayley graph $\text{Cay}(S_3, U)$ is below.



It is easy to verify that the edges drawn above correspond exactly to the set $E = \{[x, xs] : x \in S_3, s \in U\}$.

In Gauyacq [5] the notion of Cayley graphs on groups was generalised to loop structures and in [13] it was generalised to groupoids. In [13] it was in fact shown

that every graph can be represented as a Cayley graph on a groupoid. This brings us to the following definitions.

Definition 3.4. Let A be a groupoid and let $S \subset A$. If

- (i) $a \notin aS$ for any $a \in A$ and
- (ii) $a \in (as)S$ for any $a \in A$ and $s \in S$,

then S is called a *Cayley set on A* .

We note that if A is a group then the conditions in Definition 3.4 coincide with the conditions in Definition 3.1.

Now let A be a groupoid and S a Cayley set on A . Let us verify the required irreflexive and symmetric relation E . Just as with groups, we define E on A by $(x, y) \in E$ if and only if $y = xs$ where $s \in S$. If $(x, y) \in E$ then $y = xs$ for some $s \in S$, and we also have that $x = (xs)s'$ for some $s' \in S$. Therefore $ys' = (xs)s' = x$ so that $(y, x) \in E$ and E is symmetric. It is also clear that E is irreflexive since $x \notin xS$.

Naturally, Cayley graphs on groupoids are defined exactly as Cayley graphs on groups. Formally it is as follows.

Definition 3.5. Let A be a groupoid and S a Cayley set on A . The graph Γ consisting of the vertex set

$$V(\Gamma) = A$$

and edge set

$$E(\Gamma) = \{[a, as] : a \in A, s \in S\}$$

is called a *Cayley graph on A* and is denoted as $\text{Cay}(A, S)$.

In the literature, it is generally not explicitly stated that a Cayley graph is Cayley on a group. If a graph is referred to as Cayley, it is assumed to mean Cayley on a group. However, since we will be dealing with Cayley graphs on loops, we will be explicit in referring to Cayley graphs on groups.

3.2 Quasi-associativity

As alluded to previously, graphs which are Cayley on a group have been proven to be vertex-transitive, but the reverse implication does not hold. However, it has been shown that for a Cayley graph on a groupoid $\text{Cay}(A, S)$, all the structure that is required for vertex-transitivity is that A is a quasi-group and S conforms to a certain type of weak associativity called quasi-associativity. This notion of quasi-associative subsets was first introduced by Gauyacq in [5] who termed them right associative. As elsewhere [13, 14, 15, 16], we will continue to call them quasi-associative.

Definition 3.6. Let A be a groupoid and S a subset of A . If for all $x, y \in A$ we have that

$$x(yS) = (xy)S$$

then S is called a *quasi-associative* subset of A .

Proposition 3.7. [15] *Let A be a quasi-group and S a quasi-associative Cayley set on A . Then the graph $\Gamma = \text{Cay}(A, S)$ is vertex-transitive.*

Proof. For each $a \in A$ we denote by $\lambda_a : A \rightarrow A$ the left translation

$$\lambda_a(x) = ax$$

for any $x \in A$. Since A is a quasi-group, each ax is unique in A for a given $a \in A$ and any $x \in A$. Therefore, λ_a is a permutation of $V(\Gamma)$.

Every edge in Γ is of the form $[x, xs]$ for some $x \in A$ and some $s \in S$. Taking λ_a of $[x, xs]$ we get

$$\lambda_a([x, xs]) = [\lambda_a(x), \lambda_a(xs)] = [ax, a(xs)] = [ax, (ax)s']$$

for some $s' \in S$. Thus λ_a is edge-preserving and $\lambda_a \in \text{Aut Cay}(A, S)$.

Now let $x, y \in V(\Gamma)$. Since A admits left cancellation, there exists a unique a such that $ax = y$. Therefore, for any $x, y \in V(\Gamma)$, there exists some $\lambda_a \in \text{Aut Cay}(A, S)$ such that $\lambda_a(x) = y$ so that $\text{Aut } \Gamma$ acts transitively on $V(\Gamma)$. Therefore, $\Gamma = \text{Cay}(A, S)$ is vertex-transitive. \square

Remark 3.8. If A is a group then the set of left translations forms a group under composition. This is an application of Cayley's theorem. The set of left translations

has the same order as the set of vertices. Thus if A is a group, then the set of left translations acts regularly on the set of vertices.

Graphs which can be represented by a quasi-group and a quasi-associative Cayley set are called *quasi-Cayley* graphs.

The following is an example of a graph defined on a groupoid, which in fact turns out to be a loop, with a quasi-associative Cayley set.

Example 3.9. [15] Let n and r be integers such that $(n, r) = 1$ with $r^2 \equiv \pm 1 \pmod{n}$. Consider the set $M(n, r) = \mathbb{Z}_2 \times \mathbb{Z}_n$ paired with the binary operation defined by $(x, y)(x', y') = (x + x', y + r^x y')$ where r is read as modulo n . It is clear that $(0, 0)$ is the identity of $M(n, r)$. Suppose that $(x, y)(x', y') = (x, y)(x'', y'')$. This implies that

$$\begin{aligned} (x + x', y + r^x y') &= (x + x'', y + r^x y'') \\ \Rightarrow x + x' &= x + x'' \text{ and } y + r^x y' = y + r^x y'' \\ \Rightarrow x' &= x'' \text{ and } r^x y' = r^x y'' \\ \Rightarrow (x', y') &= (x'', y'') \end{aligned}$$

Suppose also that $(x', y')(x, y) = (x'', y'')(x, y)$. This implies that

$$\begin{aligned} (x' + x, y' + r^{x'} y) &= (x'' + x, y'' + r^{x''} y) \\ \Rightarrow x' + x &= x'' + x \text{ and } y' + r^{x'} y = y'' + r^{x''} y \\ \Rightarrow x' &= x'' \text{ and } y' = y'' \\ \Rightarrow (x', y') &= (x'', y'') \end{aligned}$$

Therefore, $M(n, r) = \mathbb{Z}_2 \times \mathbb{Z}_n$ is a loop.

Consider the set $S = \{(0, 1), (0, -1), (1, 0)\} \subset M(n, r)$. For any $(x, y) \in M(n, r)$ we have $(x, y)S = \{(x, y + r^x), (x, y - r^x), (x + 1, y)\}$ so that $(x, y)s \neq (x, y)$ for any $s \in S$. We also have that

$$\begin{aligned} ((x, y)(0, 1))(0, -1) &= (x, y), \\ ((x, y)(0, -1))(0, 1) &= (x, y), \text{ and} \\ ((x, y)(1, 0))(1, 0) &= (x, y) \end{aligned}$$

so that $(x, y) \in ((x, y)s)S$ for any $s \in S$. Therefore S is a Cayley set on $M(n, r)$.

For any $(x, y), (x', y') \in M(n, r)$ we have

$$\begin{aligned} & ((x, y)(x', y'))S \\ &= \{(x + x', y + r^x y')(0, 1), (x + x', y + r^x y')(0, -1), (x + x', y + r^x y')(1, 0)\} \\ &= \{(x + x', y + r^x y' + r^{x+x'}), (x + x', y + r^x y' - r^{x+x'}), (x + x' + 1, y + r^x y')\} \end{aligned}$$

and

$$\begin{aligned} & (x, y)((x', y')S) \\ &= \{(x, y)((x', y')(0, 1)), (x, y)((x', y')(0, -1)), (x, y)((x', y')(1, 0))\} \\ &= \{(x, y)(x', y' + r^{x'}), (x, y)(x', y' - r^{x'}), (x, y)(x' + 1, y')\} \\ &= \{(x + x', y + r^x(y' + r^{x'})), (x + x', y + r^x(y' - r^{x'})), (x + x' + 1, y + r^x y')\} \\ &= \{(x + x', y + r^x y' + r^x r^{x'}), (x + x', y + r^x y' - r^x r^{x'}), (x + x' + 1, y + r^x y')\} \end{aligned}$$

so that S is quasi-associative if $\{r^x r^{x'}, -r^x r^{x'}\} = \{r^{x+x'}, -r^{x+x'}\}$, which is true since $r^2 \equiv \pm 1$.

The graph $\text{Cay}(M(n, r), S)$ is defined on a loop with a quasi-associative Cayley set so that it is quasi-Cayley and vertex-transitive.

As it turns out, the class of graphs referred to in Example 3.9 is in fact isomorphic to the generalised Petersen graphs.

Proposition 3.10. [15] *Let $M(n, r)$ and S be as in Example 3.9. Then $\text{Cay}(M(n, r), S) \cong G(n, r)$.*

Proof. Define a mapping $\phi : V(G(n, r)) \longrightarrow V(\text{Cay}(M(n, r), S))$ by

$$v_i \mapsto (0, i) \text{ and } u_i \mapsto (1, i).$$

Clearly ϕ is a bijection.

We also have the following

$$\begin{aligned} \phi([v_i, v_{i+1}]) &= [((0, i), (0, i + 1))] \\ &= [(0, i), (0, i)(0, 1)] \\ \phi([u_i, u_{i+r}]) &= [((1, i), (1, i + r))] \\ &= [(1, i), (1, i)(0, 1)] \end{aligned}$$

$$\begin{aligned}\phi([v_i, u_i]) &= [((0, i), (1, i))] \\ &= [(0, i), (0, i)(1, 0)]\end{aligned}$$

so that ϕ is a bijective homomorphism.

It is clear that $|E(\text{Cay}(M(n, r), S))| = |E(G(n, r))|$ since both graphs have the same degree. Therefore, by Lemma 2.5 ϕ is an isomorphism. \square

As mentioned previously, every graph is a Cayley graph on some groupoid. In [14] it was shown that every vertex-transitive graph is a Cayley graph on a left loop with respect to a quasi-associative Cayley set. In view of this, the determination of VTNCGs translates to finding Cayley graphs on left loops with respect to quasi-associative Cayley sets which cannot be represented as Cayley graphs on groups. For our purposes, we then desire to find a suitable loop on which we can construct our vertex-transitive graphs.

3.3 Meta-Cayley graphs

Alspach and Parsons introduced a class of vertex-transitive graphs called metacirculant graphs in [2]. This class of graphs contains many VTNCGs. They are defined on the direct product of two cyclic groups, with the definition of edges involving some interaction between the groups which resembles the twisting used in defining semi-direct products of groups. These graphs were first shown to be quasi-Cayley by Gauyacq in [5]. In [16] they were generalised from cyclic groups to general groups.

It is a well-known result that if the twisting map in the semi-direct product of groups is a group homomorphism, then the semi-direct product is a group itself. For the semi-direct product to be a loop, much less structure is required for the twisting map. Left and right cancellability are maintained without any condition on the twisting map. For the existence of an identity element the twisting map need only satisfy a weak condition. That is, that the twisting map maps the identity element in the first group to the identity map of the second group. In the generalisation of metacirculant graphs from cyclic groups to general groups, only this key characteristic of the twisting map was maintained. The resultant graphs of this generalised construction are called *meta-Cayley* graphs. Meta-Cayley graphs will be suitable for our purposes.

Before we can present the definition of meta-Cayley graphs, it is necessary for us to first present some propositions and lemmas. The following proposition shows that the semi-direct product of two groups with a twisting map that satisfies the weak condition mentioned previously is indeed a loop.

Proposition 3.11. ([16]) *Let A and A' be groups. Let $Q = A \times_f A'$ be the semi-direct product of A and A' such that $f(e)$ is the identity map on A' , where e is the identity element of A . Then Q is a loop.*

Proof. Let $(a, a'), (b, b') \in Q$. Since A and A' are groups, there exists unique x and x' such that $ax = b$ and $a'x' = b'$. Also, since f_a is an automorphism of A' , there exists a unique x'' such that $f_a(x'') = x'$. Therefore, there exists a unique (x, x'') such that $(a, a')(x, x'') = (b, b')$ so that Q is a left quasi-group.

Since A and A' are groups, there exists unique y and y' such that $ya = b$ and $y'f_y(a') = b'$. Therefore, there exists a unique (y, y') such that $(y, y')(a, a') = (b, b')$ so that Q is a right quasi-group.

Furthermore, let e and e' be the identity elements of A and A' respectively. Then $(x, y)(e, e') = (xe, yf_x(e')) = (xe, ye') = (x, y)$ and $(e, e')(x, y) = (ex, e'f_e(y)) = (ex, e'y) = (x, y)$ so that (e, e') is the identity. Therefore, $Q = A \times_f A'$ is a loop. \square

Now that we know we can sufficiently find a loop by taking the semi-direct product of two groups as above, the crux of the matter with regards to constructing meta-Cayley graphs is the determination of quasi-associative Cayley sets. Before we show how we determine these meta-Cayley sets, we first need the following lemmas.

Lemma 3.12. *Let A be a loop and let U_1, U_2, \dots, U_k be quasi-associative subsets of A . Then $U := \bigcup U_i$ is quasi-associative in A .*

Proof. Let $a, b \in A$ and let $s \in U$ so that $s \in U_i$ for some i . Then $a(bs) = (ab)s'$ where $s' \in U_i$. Therefore $s' \in U$ and U is quasi-associative in A . \square

Lemma 3.13. [16] *Let A be a loop with an identity element e , and U a quasi-associative subset of A such that $e \notin U$. Then U is Cayley if $U^{-1} \subseteq U$.*

Proof. Let U be a quasi-associative subset of A with $e \notin U$ and $U^{-1} \subseteq U$. It is clear that $a \notin aU$ for any $a \in A$. For any $a \in A$ and $u \in U$, we have that $a = ae = a(uu^{-1}) = (au)u'$ for some $u' \in U$. Therefore, $a \in (au)U$ for any $a \in A$

and $u \in U$. Therefore, U is Cayley. \square

Proposition 3.14. ([16]) *Let A and A' be groups and Q be as in Proposition 3.11. For each $x \in A$, let L_x be a (possibly empty) subset of A' such that*

- (a) $e' \notin L_e$ where e and e' are the identity elements of A and A' respectively;
- (b) $L_{x^{-1}} = f_x^{-1}[L_x^{-1}]$ for any $x \in A$;
- (c) $f_a f_b[L_x] = f_{ab}[L_x]$ for any $a, b \in A$.

Let U be the subset of Q defined by

$$U := \bigcup_{x \in A} (\{x\} \times L_x).$$

Then U is a quasi-associative Cayley set on Q .

Proof. Since A and A' are groups, both $\{x\}$ and L_x are trivially quasi-associative in A and A' respectively for all $x \in A$. Let $(a, b), (a', b') \in Q$ and $(x, s) \in \{x\} \times L_x$ for some $x \in A$. Then

$$\begin{aligned} (a, b)((a', b')(x, s)) &= (a, b)(a'x, b'f_{a'}(s)) \\ &= (aa'x, bf_a(b'f_{a'}(s))) \\ &= (aa'x, bf_a(b')f_a(f_{a'}(s))) \\ &= (aa'x, bf_a(b')f_{aa'}(s')) \\ &= (aa', bf_a(b'))(x, s') \\ &= ((a, b)(a', b'))(x, s') \end{aligned}$$

for some $(x, s') \in \{x\} \times L_x$. Therefore, $\{x\} \times L_x$ is quasi-associative in Q . By Lemma 3.12, U is quasi-associative in Q .

Clearly, $(e, e') \notin U$. In view of Lemma 3.13, it is enough to show that $U^{-1} \subseteq U$. For each $x \in A$, consider the set $U_x = (\{x\} \times L_x) \cup (\{x^{-1}\} \times L_{x^{-1}})$. We have

$$(\{x\} \times L_x)^{-1} = (\{x^{-1}\} \times f_x^{-1}[L_x^{-1}]) = (\{x^{-1}\} \times L_{x^{-1}})$$

and

$$(\{x^{-1}\} \times L_{x^{-1}})^{-1} = (\{x\} \times f_{x^{-1}}^{-1}[L_{x^{-1}}^{-1}]) = (\{x\} \times L_x).$$

Hence $U_x = U_x^{-1}$ for any $x \in A$. It is also easy to see that

$$\bigcup_{x \in A} U_x = U.$$

Therefore $U = U^{-1}$ and by Lemma 3.13, U is Cayley. □

With Q and U as above, the resultant graph $\text{Cay}(Q, U)$ is then necessarily vertex-transitive. This brings us to the definition of meta-Cayley graphs.

Definition 3.15. Let A and A' be groups. Let $Q = A \times_f A'$ be the semi-direct product of A and A' such that $f(e)$ is the identity map on A' , where e is the identity element of A . For each $x \in A$, let L_x be a (possibly empty) subset of A' such that

- (a) $e' \notin L_e$ where e and e' are the identity elements of A and A' respectively;
- (b) $L_{x^{-1}} = f_x^{-1}[L_x^{-1}]$ for any $x \in A$;
- (c) $f_a f_b[L_x] = f_{ab}[L_x]$ for any $a, b \in A$.

Let U be the subset of Q defined by

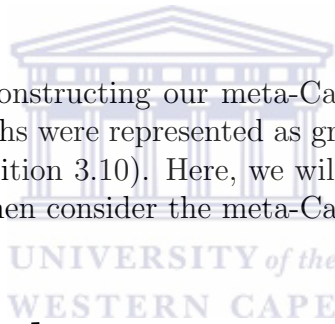
$$U := \bigcup_{x \in A} (\{x\} \times L_x).$$

We call U a *meta-Cayley* subset of Q and the graph $\text{Cay}(Q, U)$ is called a *meta-Cayley* graph.

Chapter 4

Constructing graphs on dihedral groups

In this chapter, we will be constructing our meta-Cayley graphs on loops. In [16] the generalised Petersen graphs were represented as graphs on the loop $(\mathbb{Z}_2 \times \mathbb{Z}_n, \oplus)$ (see Example 3.9 and Proposition 3.10). Here, we will consider the dihedral groups D_n in place of \mathbb{Z}_n . We will then consider the meta-Cayley graphs on these loop.



4.1 Constructing loops

In view of Proposition 3.14 the loop we will consider is the case of $A = \mathbb{Z}_2$ and $A' = D_n$. To facilitate our discussion, we will represent the dihedral group D_n as the semi-direct product of \mathbb{Z}_2 and \mathbb{Z}_n . That is, D_n as $\mathbb{Z}_2 \times_g \mathbb{Z}_n$ where $g : \mathbb{Z}_2 \rightarrow \text{Aut } \mathbb{Z}_n$ defined by $g_x(y) = (-1)^x y$. It is easily shown that $g_x \in \text{Aut } \mathbb{Z}_n$. This representation will make it easier to present the loop operation. We present the following proof that this is admissible.

Proposition 4.1. *Let $Q = \mathbb{Z}_2 \times_g \mathbb{Z}_n$ where $g : \mathbb{Z}_2 \rightarrow \text{Aut } \mathbb{Z}_n$ defined by $g_x(y) = (-1)^x y$. Then Q is isomorphic to the dihedral group D_n .*

Proof. Let $\phi : Q \rightarrow D_n$ be a mapping defined by

$$\phi : (0, i) \mapsto R_i \text{ and } \phi : (1, i) \mapsto S_i$$

where R_i and S_i represent rotations and reflections in D_n respectively. It is clear that ϕ is a bijection.

We require that the groupoid operation is preserved. There are four cases to consider:

- (i) $\phi((0, i)(0, j)) = \phi((0, i + j)) = R_{i+j} = R_i R_j = \phi((0, i))\phi((0, j))$
- (ii) $\phi((0, i)(1, j)) = \phi((1, i + j)) = S_{i+j} = R_i S_j = \phi((0, i))\phi((1, j))$
- (iii) $\phi((1, i)(0, j)) = \phi((1, i - j)) = S_{i-j} = S_i R_j = \phi((1, i))\phi((0, j))$
- (iv) $\phi((1, i)(1, j)) = \phi((0, i - j)) = R_{i-j} = S_i S_j = \phi((1, i))\phi((1, j))$

Therefore ϕ is an isomorphism and $Q \cong D_n$. □

In view of Proposition 4.1 and Proposition 3.14, we will consider the case of $A = \mathbb{Z}_2$ and $A' = \mathbb{Z}_2 \times_g \mathbb{Z}_n$. Taking the semi-direct product of these two groups we have $\mathbb{Z}_2 \times_f (\mathbb{Z}_2 \times_g \mathbb{Z}_n)$.

As we can see, we have to deal with two twisting maps: g to define dihedral groups on the set $\mathbb{Z}_2 \times \mathbb{Z}_n$ and f to define loops. At this point, we are yet to define $f : \mathbb{Z}_2 \rightarrow \text{Aut } \mathbb{Z}_2 \times_g \mathbb{Z}_n$. Note that any automorphism of $\mathbb{Z}_2 \times_g \mathbb{Z}_n$ will respect the first co-ordinate since any automorphism on D_n preserves both the set of reflections and rotations (see [12]). We now define f by $f_x(y, z) = (y, r^x z)$ where $r \in \mathbb{Z}_n$ such that $(n, r) = 1$ and prove that this mapping suffices as an automorphism of $\mathbb{Z}_2 \times_g \mathbb{Z}_n$.

Proposition 4.2. *Let $r \in \mathbb{Z}_n$ such that $(n, r) = 1$. Let $f_x : \mathbb{Z}_2 \times_g \mathbb{Z}_n \rightarrow \mathbb{Z}_2 \times_g \mathbb{Z}_n$ be a mapping defined by $f_x(y, z) = (y, r^x z)$ where $x \in \mathbb{Z}_2$. Then f is an automorphism.*

Proof. Since $(n, r) = 1$ it is clear that f is onto. f_x is thus a bijection.

Now,

$$\begin{aligned}
 f_x((y, z)(y', z')) &= f_x((y + y', z + (-1)^y z')) \\
 &= (y + y', r^x(z + (-1)^y z')) \\
 &= (y + y', r^x z + r^x (-1)^y z') \\
 &= (y, r^x z)(y', r^x z')
 \end{aligned}$$

$$= f_x((y, z))f_x((y', z'))$$

Therefore f_x is an automorphism. \square

Our groupoid is therefore fully defined as $Q(n, r) = \mathbb{Z}_2 \times_f (\mathbb{Z}_2 \times_g \mathbb{Z}_n)$ with $r \in \mathbb{Z}_n$, $(n, r) = 1$ and the binary operation defined by

$$\begin{aligned} (x, (y, z))(x', (y', z')) &= (x + x', (y, z)f_x(y', z')) \\ &= (x + x', (y, z)(y', r^x z')) \\ &= (x + x', (y + y', z + g_y(r^x z'))) \\ &= (x + x', (y + y', z + (-1)^y r^x z')) \end{aligned}$$

with addition modulo 2 in the first and second co-ordinate and modulo n in the third co-ordinate. Henceforth we will not explicitly state this. For brevity, we denote $(x, (y, z))$ as (x, y, z) .

As mentioned, we require that our groupoid is in fact a loop. We apply Proposition 3.11 in the specific context of our groupoid $Q(n, r)$. We also go further and identify the identity element.

Proposition 4.3. *Let $r \in \mathbb{Z}_n$ such that $(n, r) = 1$. Let $Q(n, r) = \mathbb{Z}_2 \times_f (\mathbb{Z}_2 \times_g \mathbb{Z}_n)$ with binary operation defined by $(x, y, z)(x', y', z') = (x + x', y + y', z + (-1)^y r^x(z'))$. Then $Q(n, r)$ is a loop with $(0, 0, 0)$ as the identity element.*

Proof. We have that

$$f_0(y, z) = (y, r^0 z) = (y, z)$$

so that f_0 is the identity map on $\mathbb{Z}_2 \times_g \mathbb{Z}_n$. By Proposition 3.11, $Q(n, r)$ is a loop.

For any $(x, y, z) \in Q(n, r)$, we have

$$(x, y, z)(0, 0, 0) = (x + 0, y + 0, z + (-1)^y r^x(0)) = (x, y, z)$$

and

$$(0, 0, 0)(x, y, z) = (0 + x, 0 + y, 0 + (-1)^0 r^0(z)) = (x, y, z).$$

Therefore, $(0, 0, 0)$ as the identity element. \square

Even with this weak twisting map f , it may happen that $Q(n, r)$ is a group, and not merely just a loop. In pursuit of VTNCGs, it is not beneficial to consider cases where $Q(n, r)$ is a group since graphs on $Q(n, r)$ would be by definition Cayley on groups. Therefore, before we determine possible and suitable meta-Cayley subsets, we need to characterise the conditions under which $Q(n, r)$ is a group.

Proposition 4.4. *Let $r \in \mathbb{Z}_n$ such that $(n, r) = 1$. Let $Q(n, r) = \mathbb{Z}_2 \times_f (\mathbb{Z}_2 \times_g \mathbb{Z}_n)$ with binary operation defined by $(x, y, z)(x', y', z') = (x + x', y + y', z + (-1)^y r^x(z'))$. Then $Q(n, r)$ is a group if and only if $r^2 \equiv 1 \pmod{n}$.*

Proof. In view of Proposition 4.3, it is enough to show associativity. Now,

$$\begin{aligned} & ((x_0, x_1, x_2)(y_0, y_1, y_2))(z_0, z_1, z_2) \\ &= (x_0 + y_0, x_1 + y_1, x_2 + (-1)^{x_1} r^{x_0} y_2)(z_0, z_1, z_2) \\ &= (x_0 + y_0 + z_0, x_1 + y_1 + z_1, x_2 + (-1)^{x_1} r^{x_0} y_2 + (-1)^{x_1+y_1} r^{x_0+y_0} z_2) \end{aligned}$$

and

$$\begin{aligned} & (x_0, x_1, x_2)((y_0, y_1, y_2)(z_0, z_1, z_2)) \\ &= (x_0, x_1, x_2)(y_0 + z_0, y_1 + z_1, y_2 + (-1)^{y_1} r^{y_0} z_2) \\ &= (x_0 + y_0 + z_0, x_1 + y_1 + z_1, x_2 + (-1)^{x_1} r^{x_0} (y_2 + (-1)^{y_1} r^{y_0} z_2)). \end{aligned}$$

The first and second co-ordinate are always equal. So for associativity to hold, it is required that we have

$$\begin{aligned} & x_2 + (-1)^{x_1} r^{x_0} (y_2) + (-1)^{x_1+y_1} r^{x_0+y_0} z_2 = x_2 + (-1)^{x_1} r^{x_0} (y_2 + (-1)^{y_1} r^{y_0} z_2) \\ \Leftrightarrow & (-1)^{x_1} r^{x_0} y_2 + (-1)^{x_1+y_1} r^{x_0+y_0} z_2 = (-1)^{x_1} r^{x_0} y_2 + (-1)^{x_1} r^{x_0} (-1)^{y_1} r^{y_0} z_2 \\ \Leftrightarrow & (-1)^{x_1+y_1} r^{x_0+y_0} z_2 = (-1)^{x_1} r^{x_0} (-1)^{y_1} r^{y_0} z_2. \end{aligned}$$

Since $x_0 + y_0 \in \mathbb{Z}_2$, this holds without any conditions except when $x_0 = y_0 = 1$. In this case, it holds if and only if $r^2 \equiv 1 \pmod{n}$. \square

In view of the above discussion and Proposition 4.4, we do not consider graphs where $r^2 \equiv 1 \pmod{n}$.

4.2 Meta-Cayley subsets

As mentioned, central to the construction of the graphs is the identification of quasi-associative meta-Cayley sets. Naturally the form of these sets is dependant on r . There is no general form of these sets unless we restrict r in a particular way. It turns out that when $r^2 \equiv -1 \pmod{n}$ we may determine a general form of the meta-Cayley subsets without too much difficulty, for any n . For that reason, we restrict our consideration of $Q(n, r)$ to where $r^2 \equiv -1 \pmod{n}$. By applying the conditions in Definition 3.15 in our context, the following lemma reveals the possible forms of the meta-Cayley subsets of $Q(n, r)$.

Lemma 4.5. *Let $r \in \mathbb{Z}_n$ such that $(n, r) = 1$ and $r^2 \equiv -1 \pmod{n}$. Let $Q(n, r) = \mathbb{Z}_2 \times_f (\mathbb{Z}_2 \times_g \mathbb{Z}_n)$ with binary operation defined by $(x, y, z)(x', y', z') = (x + x', y + y', z + (-1)^y r^x(z'))$ and let U be a subset of $Q(n, r)$. Then U is a meta-Cayley subset of $Q(n, r)$ if and only if*

- (i) $(0, 0, 0) \notin U$,
- (ii) $(0, 0, i) \in U$ implies $(0, 0, -i) \in U$,
- (iii) $(0, 1, i) \in U$ implies $(0, 1, -i) \in U$,
- (iv) $(1, 0, i) \in U$ implies $(1, 0, -i), (1, 0, ri), (1, 0, -ri) \in U$, and
- (v) $(1, 1, i) \in U$ implies $(1, 1, -i), (1, 1, ri), (1, 1, -ri) \in U$.

Proof. We require that U is constructed as in Definition 3.15. Thus we require sets L_0 and L_1 that satisfies the conditions (a), (b) and (c) of Definition 3.15 .

(a) By (i), $(0, 0, 0) \notin U$.

(b) $L_{0^{-1}} = f_0^{-1}[L_0^{-1}]$ which implies that $L_0 = L_0^{-1}$. Now $(0, i)^{-1} = (0, -i)$ and $(1, i)^{-1} = (1, i)$ in $\mathbb{Z}_2 \times_g \mathbb{Z}_n$. Therefore $(0, i) \in L_0$ implies $(0, -i) \in L_0$, which means that $(0, 0, i) \in U$ implies $(0, 0, -i) \in U$.

$L_{1^{-1}} = f_1^{-1}[L_1^{-1}]^{-1}$ which implies that $L_1 = f_1^{-1}[L_1]^{-1}$. Now $f_1^{-1}((0, i)^{-1}) = f_1^{-1}((0, -i)) = (0, ri)$. Therefore $(0, i) \in L_1$ implies $(0, ri), (0, -i), (0, -ri) \in L_1$. Similarly, $(1, i) \in L_1$ implies $(1, -ri), (1, -i), (1, ri) \in L_1$. This means that $(1, 0, i) \in U$ implies $(1, 0, ri), (1, 0, -i), (1, 0, -ri) \in U$ and $(1, 1, i) \in U$ implies $(1, 1, ri), (1, 1, -i), (1, 1, -ri) \in U$.

- (c) Since f_0 is identity map we need only consider when $a = b = 1$. In which case, we have $f_1 f_1[L_x] = f_0[L_x]$. Therefore $(j, i) \in L_x$ implies that $f_1 f_1(j, i) = (j, -i) \in L_x$. This means that $(j, i) \in U$ implies $(j, -i) \in U$. In particular, $(0, 1, i) \in U$ implies $(0, 1, -i) \in U$.

(a) corresponds to condition (i), (b) corresponds to conditions (ii),(iv) and (v), and (c) corresponds to condition (iii). \square

It is not difficult to see that for disconnected vertex-transitive graphs, each component is necessarily vertex-transitive and they are isomorphic to each other. Further, for a disconnected VTNCG, each component is itself a connected VTNCG. For this reason, in the pursuit of VTNCGs, we generally only consider the connected case.

Now that we have identified possibilities of meta-Cayley subsets in Lemma 4.5, let us discuss the nature of the adjacencies that may ensue on these graphs, given particular elements in the meta-Cayley subset. The reason why we want to discuss this matter is to better understand when our graphs are necessarily connected based on the form of the meta-Cayley sets. Also, we would like to be able to identify cycles which may exist. This in turn will assist in determining automorphisms of the graphs.

In order to facilitate the discussion about possible adjacencies, we partition $V(\text{Cay}(Q(n, r), U))$ into four natural sets as follows. Define

$$V_{00} := \{(0, 0, i) : i \in \mathbb{Z}_n\}; \quad (4.1)$$

$$V_{10} := \{(1, 0, i) : i \in \mathbb{Z}_n\}; \quad (4.2)$$

$$V_{11} := \{(1, 1, i) : i \in \mathbb{Z}_n\}; \quad (4.3)$$

$$V_{01} := \{(0, 1, i) : i \in \mathbb{Z}_n\}. \quad (4.4)$$

Now, we note that we have the following admissible adjacencies in relation to this partition.

Lemma 4.6. *Let $Q(n, r)$ be as in Lemma 4.5 and let U be a meta-Cayley subset of $Q(n, r)$. Let $V(\text{Cay}(Q(n, r), U))$ be partitioned as in (4.1) to (4.4). In $\text{Cay}(Q(n, r), U)$*

- (a) U contains element(s) of the form $(1, 0, i)$ if and only if
 (i) every vertex in V_{00} is adjacent to some vertex in V_{10} , and

- (ii) every vertex in V_{01} is adjacent to some vertex in V_{11} .
- (b) U contains element(s) of the form $(1, 1, i)$ if and only if
 - (i) every vertex in V_{00} is adjacent to some vertex in V_{11} , and
 - (ii) every vertex in V_{01} is adjacent to some vertex in V_{10} .
- (c) U contains element(s) of the form $(0, 1, i)$ if and only if
 - (i) every vertex in V_{00} is adjacent to some vertex in V_{01} , and
 - (ii) every vertex in V_{10} is adjacent to some vertex in V_{11} .
- (d) U contains element(s) of the form $(0, 0, i)$ if and only if every vertex is adjacent to some vertex in the same set in the partition.

Proof. We present a proof for (a)(i). A similar argument applies for the rest of the lemma.

Let U contain an element $(1, 0, i)$. For every vertex $(0, 0, j)$ in V_{00} we have an edge $[(0, 0, j), (0, 0, j)(1, 0, i)] = [(0, 0, j), (1, 0, j + i)]$. Thus every vertex in V_{00} is adjacent to some vertex V_{10} .

If a vertex $(0, 0, j)$ in V_{00} is adjacent to some vertex $(1, 0, j + i)$ in V_{10} then there exists an element (x, y, z) in U such that $[(0, 0, j), (0, 0, j)(x, y, z)] = [(0, 0, j), (1, 0, j + i)]$. We then have that $(x, y, z) = (1, 0, i)$. \square

Figure 4.1 illustrates points (a)(i), (b)(i), (c)(i) and (d) of Lemma 4.6. The figure displays the edges involving the vertex $(0, 0, j)$ interacting with elements $(0, 0, i), (1, 0, i), (0, 1, i)$ and $(1, 1, i)$ from the Cayley set.

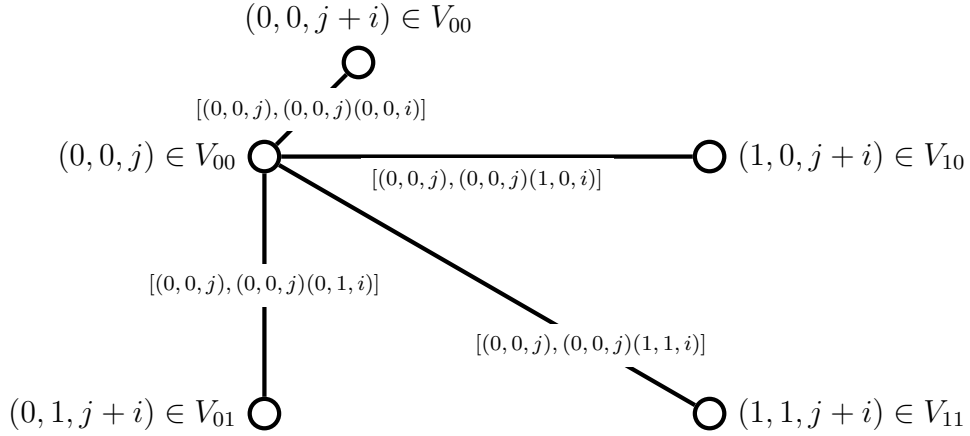


Figure 4.1: Adjacencies as a result of particular elements in U .

In view of Lemma 4.6, we have the following, which presents conditions for connectedness of $\text{Cay}(Q(n, r), U)$ for a particular form of U . Note that U in the corollary is chosen in view of Lemma 4.5 as well.

Corollary 4.7. *Let n be odd and $i, j \in \mathbb{Z}_n$ such that $(i, n) = 1$. Let*

$$U = \{(1, 0, j), (1, 0, -j), (1, 0, rj), (1, 0, -rj), (0, 1, i), (0, 1, -i)\}$$

be a meta-Cayley subset of $Q(n, r)$ where $Q(n, r)$ is as in Lemma 4.5. Then $\text{Cay}(Q(n, r), U)$ is connected.

Proof. Since $(i, n) = 1$, the sequence $0, i, 2i, 3i, \dots, (n-1)i$ covers the numbers $0, 1, 2, 3, \dots, (n-1)$. The same can be said for the sequence $0, ir, 2ir, 3ir, \dots, (n-1)ir$. We trace a path starting at $(0, 0, 0)$ and operating with the element $(0, 1, i)$. We get $(0, 0, 0), (0, 1, i), (0, 0, 2i), (0, 1, 3i), \dots$. Since n is odd, we will eventually cover n vertices to get to $(0, 1, 0)$. Continuing like this, we would cover another n vertices to get back to $(0, 0, 0)$. Therefore, the cycle $C_0 = (0, 0, 0), (0, 1, i), (0, 0, 2i), \dots, (0, 0, 0)$ covers all the vertices in V_{00} and V_{01} . Similarly, the cycle $C_1 = (1, 0, 0), (1, 1, ir), (1, 0, 2ir), \dots, (1, 0, 0)$ covers all the vertices in V_{10} and V_{11} . U contains elements of the form $(1, 0, j')$, therefore there are vertices in $V_{00} \cup V_{01}$ which are adjacent to vertices in $V_{10} \cup V_{11}$. Consequently, $\text{Cay}(Q, U)$ is connected. \square

Remark 4.8. If $j = 0$, then the set U above is in fact of the form

$$U = \{(1, 0, 0), (0, 1, i), (0, 1, -i)\}.$$

Nowhere in the above proof is it required that $j \neq 0$, therefore the connectedness still holds in this case.

The following propositions show that U as in Corollary 4.7 covers a range of isomorphism types of $\text{Cay}(Q(n, r), U)$ for different U , and thus serves as a suitable Cayley set for our purposes.

Proposition 4.9. *Let n be odd and $i, j \in \mathbb{Z}_n$. Let U be as in Corollary 4.7 and $Q(n, r)$ be as in Lemma 4.5. Let*

$$U' = \{(1, 1, j), (1, 1, -j), (1, 1, rj), (1, 1, -rj), (0, 1, i), (0, 1, -i)\}.$$

Then $\text{Cay}(Q(n, r), U) \cong \text{Cay}(Q(n, r), U')$.

Proof. Define a mapping $\phi : V(\text{Cay}(Q(n, r), U)) \rightarrow V(\text{Cay}(Q(n, r), U'))$ by

$$\phi : (x, y, z) \mapsto (x, y + x, z).$$

Note that ϕ fixes the elements in $V_{00} \cup V_{01}$ and swaps the elements in V_{10} with the elements in V_{11} . It is clear that ϕ is a bijection.

Now for any vertex $(0, y, z)$ in V , we have the following

$$\begin{aligned} \phi([(0, y, z), (0, y, z)(1, 0, \pm j)]) &= \phi([(0, y, z), (1, y, z + r^0(-1)^y(\pm j))]) \\ &= [(0, y, z), (1, y + 1, z + r^0(-1)^y(\pm j))] \\ &= [(0, y, z), (0, y, z)(1, 1, \pm j)] \end{aligned}$$

$$\begin{aligned} \phi([(0, y, z), (0, y, z)(1, 0, \pm rj)]) &= \phi([(0, y, z), (1, y, z + r^0(-1)^y(\pm rj))]) \\ &= [(0, y, z), (1, y + 1, z + r^0(-1)^y(\pm rj))] \\ &= [(0, y, z), (0, y, z)(1, 1, \pm rj)] \end{aligned}$$

$$\begin{aligned} \phi([(0, y, z), (0, y, z)(0, 1, \pm i)]) &= \phi([(0, y, z), (0, y + 1, z + r^0(-1)^y(\pm i))]) \\ &= [(0, y, z), (0, y + 1, z + r^0(-1)^y(\pm i))] \\ &= [(0, y, z), (0, y, z)(0, 1, \pm i)] \end{aligned}$$

For any vertex $(1, y, z)$ in V , we have the following

$$\begin{aligned}\phi([(1, y, z), (1, y, z)(1, 0, \pm j)]) &= \phi([(1, y, z), (0, y, z + r(-1)^y(\pm j))]) \\ &= [(1, y + 1, z), (0, y, z + r(-1)^y(\pm j))] \\ &= [(1, y + 1, z), (0, y, z + r(-1)^{y+1}(\pm j))] \\ &= [(1, y + 1, z), (1, y + 1, z)(1, 1, \pm j)]\end{aligned}$$

$$\begin{aligned}\phi([(1, y, z), (1, y, z)(1, 0, \pm rj)]) &= \phi([(1, y, z), (0, y, z + r(-1)^y(\pm rj))]) \\ &= [(1, y + 1, z), (0, y, z + r(-1)^y(\pm rj))] \\ &= [(1, y + 1, z), (0, y, z + r(-1)^{y+1}(\pm rj))] \\ &= [(1, y + 1, z), (1, y + 1, z)(1, 1, \pm rj)]\end{aligned}$$

$$\begin{aligned}\phi([(1, y, z), (1, y, z)(0, 1, \pm i)]) &= \phi([(1, y, z), (1, y + 1, z + r(-1)^y(\pm i))]) \\ &= [(1, y + 1, z), (1, y, z + r(-1)^y(\pm i))] \\ &= [(1, y + 1, z), (1, y + 1, z)(0, 1, \pm i)]\end{aligned}$$

Thus ϕ is a bijective homomorphism.

Both graphs have the same degree and vertex set. Therefore $|E(\text{Cay}(Q(n, r), U))| = |E(\text{Cay}(Q(n, r), U'))|$ and by Lemma 2.5, ϕ is an isomorphism. \square

The following proposition allows us to simplify our consideration of U .

Proposition 4.10. *Let n be odd and $i, j \in \mathbb{Z}_n$ such that $i = j \neq 0$ and $(i, n) = 1$. Let U be as in Corollary 4.7 and $Q(n, r)$ be as in Lemma 4.5. Let*

$$U' = \{(1, 0, 1), (1, 0, -1), (1, 0, r), (1, 0, -r), (0, 1, 1), (0, 1, -1)\}.$$

Then $\text{Cay}(Q(n, r), U) \cong \text{Cay}(Q(n, r), U')$.

Proof. Define a mapping $\phi : V(\text{Cay}(Q(n, r), U')) \rightarrow V(\text{Cay}(Q(n, r), U))$ by

$$\phi : (x, y, z) \mapsto (x, y, zi).$$

Since $(i, n) = 1$ we have that ϕ is a bijection.

For any vertex (x, y, z) in V , we have the following

$$\phi([(x, y, z), (x, y, z)(1, 0, 1)]) = \phi([(x, y, z), (x + 1, y, z + r^x(-1)^y(1))])$$

$$\begin{aligned}
&= [(x, y, zi), (x + 1, y, zi + r^x(-1)^y(i))] \\
&= [(x, y, zi), (x, y, zi)(1, 0, i)]
\end{aligned}$$

$$\begin{aligned}
\phi([(x, y, z), (x, y, z)(1, 0, -1)]) &= \phi([(x, y, z), (x + 1, y, z + r^x(-1)^y(-1))]) \\
&= [(x, y, zi), (x + 1, y, zi + r^x(-1)^y(-i))] \\
&= [(x, y, zi), (x, y, zi)(1, 0, -i)]
\end{aligned}$$

$$\begin{aligned}
\phi([(x, y, z), (x, y, z)(1, 0, r)]) &= \phi([(x, y, z), (x + 1, y, z + r^x(-1)^y(r))]) \\
&= [(x, y, zi), (x + 1, y, zi + r^x(-1)^y(ri))] \\
&= [(x, y, zi), (x, y, zi)(1, 0, ri)]
\end{aligned}$$

$$\begin{aligned}
\phi([(x, y, z), (x, y, z)(1, 0, -r)]) &= \phi([(x, y, z), (x + 1, y, z + r^x(-1)^y(-r))]) \\
&= [(x, y, zi), (x + 1, y, zi + r^x(-1)^y(-ri))] \\
&= [(x, y, zi), (x, y, zi)(1, 0, -ri)]
\end{aligned}$$

$$\begin{aligned}
\phi([(x, y, z), (x, y, z)(0, 1, 1)]) &= \phi([(x, y, z), (x, y + 1, z + r^x(-1)^y(1))]) \\
&= [(x, y, zi), (x, y + 1, zi + r^x(-1)^y(i))] \\
&= [(x, y, zi), (x, y, zi)(0, 1, i)]
\end{aligned}$$

$$\begin{aligned}
\phi([(x, y, z), (x, y, z)(0, 1, -1)]) &= \phi([(x, y, z), (x, y + 1, z + r^x(-1)^y(-1))]) \\
&= [(x, y, zi), (x, y + 1, zi + r^x(-1)^y(-i))] \\
&= [(x, y, zi), (x, y, zi)(0, 1, -i)]
\end{aligned}$$

Thus ϕ is a bijective homomorphism.

Both graphs have the same degree and vertex set. Therefore $|E(\text{Cay}(Q(n, r), U))| = |E(\text{Cay}(Q(n, r), U'))|$ and by Lemma 2.5, ϕ is an isomorphism. \square

In view of Remark 4.8, we also have the following isomorphism which is similar to the isomorphism in Proposition 4.10.

Proposition 4.11. *Let n be odd and $i \in \mathbb{Z}_n$ such that $(i, n) = 1$. Let U be as in Remark 4.8, $Q(n, r)$ be as in Lemma 4.5 and*

$$U' = \{(1, 0, 0), (0, 1, 1), (0, 1, -1)\}.$$

Then $\text{Cay}(Q(n, r), U) \cong \text{Cay}(Q(n, r), U')$.

Proof. Define a mapping $\phi : V(\text{Cay}(Q(n, r), U')) \longrightarrow V(\text{Cay}(Q(n, r), U))$ by

$$\phi : (x, y, z) \mapsto (x, y, zi).$$

Since $(i, n) = 1$ we have that ϕ is a bijection.

For any vertex (x, y, z) in V , we have the following

$$\begin{aligned} \phi([(x, y, z), (x, y, z)(1, 0, 0)]) &= \phi([(x, y, z), (x + 1, y, z)]) \\ &= [(x, y, zi), (x + 1, y, zi)] \\ &= [(x, y, zi), (x, y, zi)(1, 0, 0)] \end{aligned}$$

$$\begin{aligned} \phi([(x, y, z), (x, y, z)(0, 1, 1)]) &= \phi([(x, y, z), (x, y + 1, z + r^x(-1)^y(1))]) \\ &= [(x, y, zi), (x, y + 1, zi + r^x(-1)^y(i))] \\ &= [(x, y, zi), (x, y, zi)(0, 1, i)] \end{aligned}$$

$$\begin{aligned} \phi([(x, y, z), (x, y, z)(0, 1, -1)]) &= \phi([(x, y, z), (x, y + 1, z + r^x(-1)^y(-1))]) \\ &= [(x, y, zi), (x, y + 1, zi + r^x(-1)^y(-i))] \\ &= [(x, y, zi), (x, y, zi)(0, 1, -i)] \end{aligned}$$

Thus ϕ is a bijective homomorphism.

Both graphs have the same degree and vertex set. Therefore $|E(\text{Cay}(Q(n, r), U))| = |E(\text{Cay}(Q(n, r), U'))|$ and by Lemma 2.5, ϕ is an isomorphism. \square

In view of the preceding discussion, henceforth we will only consider the following two forms of U . That is,

$$U_1 = \{(1, 0, 0), (0, 1, 1), (0, 1, -1)\},$$

and

$$U_2 = \{(1, 0, 1), (1, 0, -1), (1, 0, r), (1, 0, -r), (0, 1, 1), (0, 1, -1)\}.$$

Further to this, we will also only consider $\text{Cay}(Q(n, r), U)$ where n is odd, $(n, r) = 1$ and $r^2 \equiv -1 \pmod{n}$.

We present the following examples of our graphs. Figure 4.2 is the graph $\text{Cay}(Q(5, 2), U_1)$ and Figure 4.3 is the graph $\text{Cay}(Q(5, 2), U_2)$.

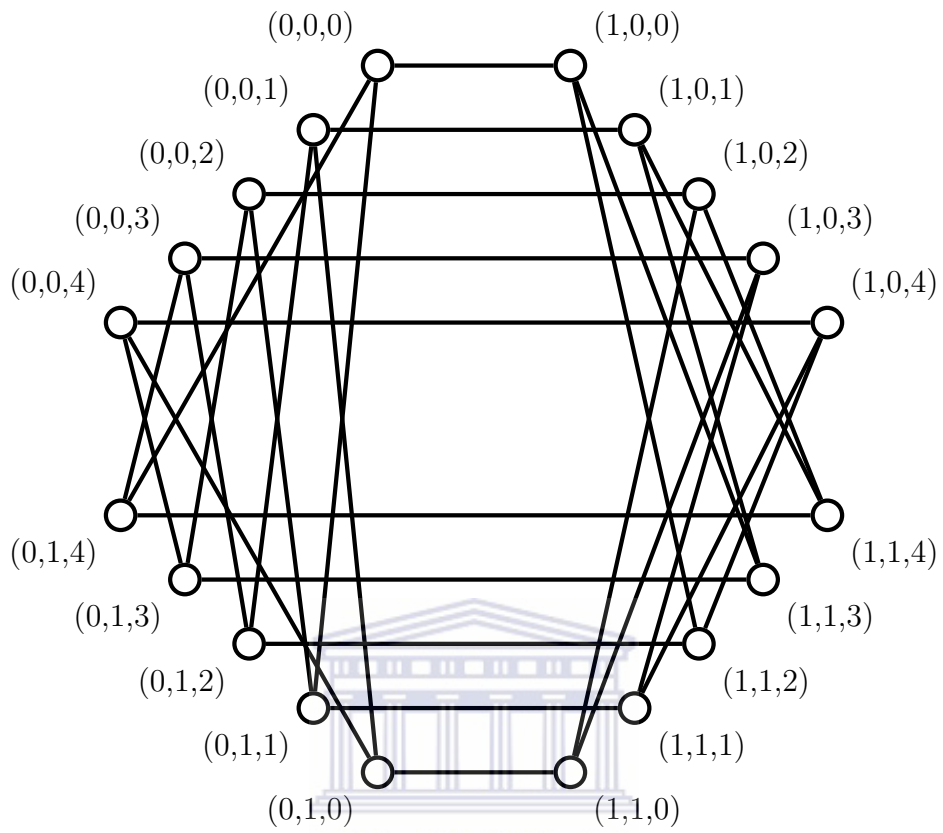


Figure 4.2: $\text{Cay}(Q(5,2), U_1)$.

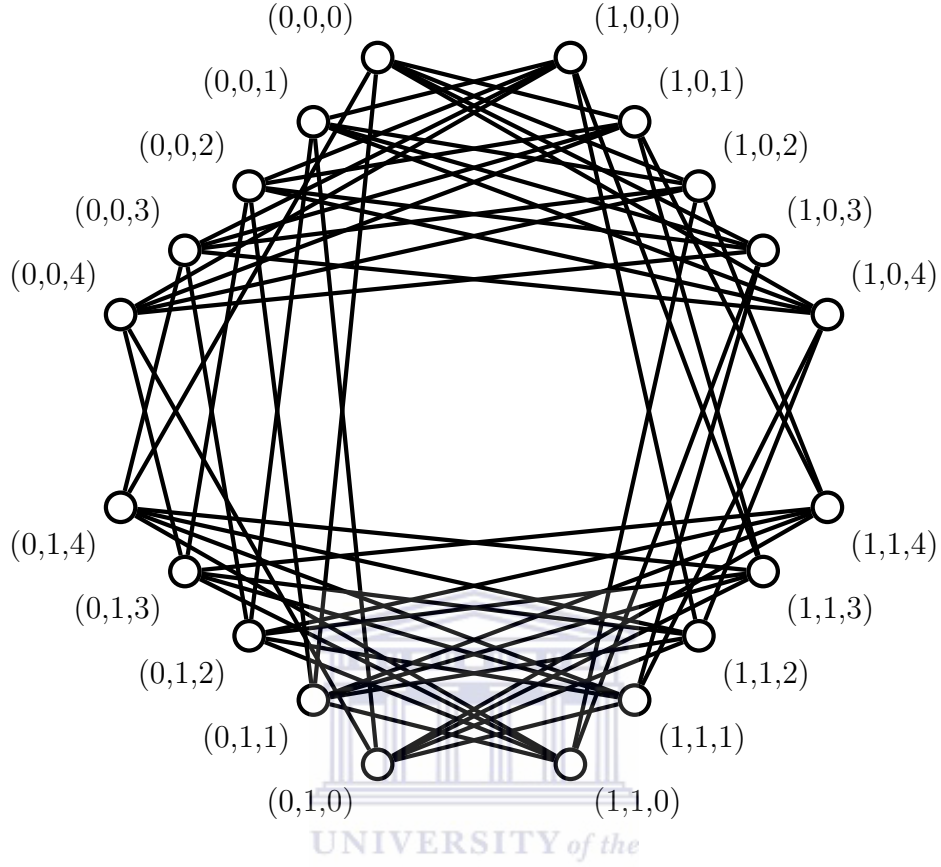


Figure 4.3: $\text{Cay}(Q(5, 2), U_2)$.

For brevity, we will define notations for our graphs to be used henceforth.

Definition 4.12. Let $Q(n, r) = \mathbb{Z}_2 \times_f (\mathbb{Z}_2 \times_g \mathbb{Z}_n)$ be a loop with binary operation $(x, y, z)(x', y', z') = (x + x', y + y', z + (-1)^{y r^x}(z'))$ where n, r are integers such that $r \in \mathbb{Z}_n$, n is odd, $(n, r) = 1$ and $r^2 \equiv -1 \pmod{n}$. Let

$$U_1 = \{(1, 0, 0), (0, 1, 1), (0, 1, -1)\}$$

and

$$U_2 = \{(1, 0, 1), (1, 0, -1), (1, 0, r), (1, 0, -r), (0, 1, 1), (0, 1, -1)\}.$$

We denote the graph $\text{Cay}(Q(n, r), U_1)$ as $\Omega(n, r)$ and the graph $\text{Cay}(Q(n, r), U_2)$ as $\Gamma(n, r)$.

Remark 4.13. U_1 and U_2 are confirmed meta-Cayley subsets of $Q(n, r)$ so that the graphs $\Omega(n, r)$ and $\Gamma(n, r)$ are meta-Cayley graphs and thus, crucially, vertex-transitive.

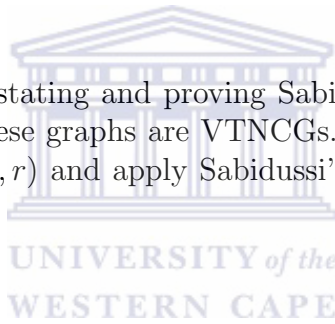
Our classes of vertex-transitive graphs now sufficiently defined, in the following chapter we determine the automorphism groups and prove non-Cayleyness on groups.



Chapter 5

Automorphism groups and non-Cayleyness

In this chapter we begin by stating and proving Sabidussi's theorem. We consider $\Omega(n, r)$ and conclude that these graphs are VTNCGs. We then fully determine the automorphism groups of $\Gamma(n, r)$ and apply Sabidussi's theorem, proving that these graphs are also VTNCGs.



5.1 Sabidussi's theorem

We present the statement and proof of Sabidussi's theorem.

Theorem 5.1. ([19]) *A graph is a Cayley graph on a group if and only if the automorphism group contains a subgroup which acts regularly on the vertex set.*

Proof. Let G be a group and S a Cayley set on G . Let $\Gamma = \text{Cay}(G, S)$. For each $g \in G$ we denote by $\lambda_g : G \rightarrow G$ the left translation

$$\lambda_g(x) = gx$$

for any $x \in G$. Each gx is unique in G for a given $g \in G$ and any $x \in G$. Therefore, λ_g is a permutation of $V(\Gamma)$.

Every edge in Γ is of the form $[x, xs]$ for some $x \in G$ and some $s \in S$. Taking λ_g of $[x, xs]$ we get

$$\lambda_g([x, xs]) = [\lambda_g(x), \lambda_g(xs)] = [gx, g(xs)] = [gx, (gx)s].$$

Thus λ_g is edge-preserving and $\lambda_g \in \text{Aut } \Gamma$.

Now define the set $\Lambda_G \subset \text{Aut } \Gamma$ as

$$\Lambda_G := \{\lambda_g : g \in G\},$$

the set of all left translations. It is clear that Λ_G has the same order as $V(\Gamma)$. Let $x, y \in V(\Gamma)$ and let $g = yx^{-1}$. We then have that $\lambda_g(x) = yx^{-1}x = y$. Therefore, for any $x, y \in V(\Gamma)$, there exists some $\lambda_g \in \Lambda_G$ such that $\lambda_g(x) = y$ so that Λ_G acts regularly on $V(\Gamma)$.

Lastly, as mentioned in Remark 3.8, by application of Cayley's theorem we have that Λ_G forms a group under composition.

Let Γ be a graph and let H be a subgroup of $\text{Aut } \Gamma$ which acts regularly on $V(\Gamma)$. Fix $u \in V(\Gamma)$. Define the set $S_u \subset H$ as

$$S_u := \{\sigma \in H : [\sigma(u), u] \in E(\Gamma)\}.$$

Now the identity map is not contained in S_u , since $[u, u] \notin E(\Gamma)$. Also, for any $\sigma \in S_u$, we have that

$$\sigma^{-1}[\sigma(u), u] = [\sigma^{-1}(\sigma(u)), \sigma^{-1}(u)] = [u, \sigma^{-1}(u)]$$

so that $\sigma^{-1} \in S_u$. Therefore, S_u is a Cayley subset of H .

Let $\Gamma_H = \text{Cay}(H, S_u)$. We claim that $\Gamma_H \cong \Gamma$. We fix $u \in V(\Gamma)$ and define a mapping $\phi : V(\Gamma_H) \rightarrow V(\Gamma)$ by $\phi : h \mapsto h(u)$. Suppose that $h(u) = h'(u)$ for some $h, h' \in H$. Let 1_H be the identity element of H . We have that

$$\begin{aligned} h(u) &= h'(u) \\ \Rightarrow u &= h^{-1}h'(u) \\ \Rightarrow h^{-1}h' &= 1_H \\ \Rightarrow h &= h' \end{aligned}$$

so that ϕ is a bijection.

For any $\sigma \in S_u$, $[h, h\sigma] \in E(\Gamma_H)$. We have

$$\phi([h, h\sigma]) = [h(u), h(\sigma(u))] = h([u, \sigma(u)]) \in E(\Gamma)$$

so that ϕ is a bijective homomorphism. Also, by regularity of H , every edge in $E(\Gamma)$ can be written as $h([u, \sigma(u)])$ for some $\sigma \in S_u$ and some $h \in H$. We then have

$$\phi^{-1}(h([u, \sigma(u)])) = \phi^{-1}([h(u), h(\sigma(u))]) = [u, \sigma(u)] \in E(\Gamma_H)$$

so that ϕ^{-1} is a homomorphism and ϕ is an isomorphism.

Therefore $\Gamma_H \cong \Gamma$ so that Γ is a Cayley graph on a group. \square

5.2 The graphs $\Omega(n, r)$

In Example 3.9 we presented a class of groupoid graphs with a quasi-associative Cayley set $\text{Cay}(M(n, r), S)$, which were in fact isomorphic to the generalised Petersen graphs. The Cayley set in question was of the form $S = \{(0, 1), (0, -1), (1, 0)\}$, which seems to resemble the form of $U_1 = \{(0, 1, 1), (0, 1, -1), (1, 0, 0)\}$. As it turns out, the graphs $\Omega(n, r)$ are in fact isomorphic to $\text{Cay}(M(n, r), S)$. Before we present the isomorphic relation, we recall that the class of graphs $\Omega(n, r)$ is restricted to where n is odd, $(n, r) = 1$ and $r^2 \equiv -1 \pmod{n}$. In $\text{Cay}(M(n, r), S)$, n is not necessarily odd.

Proposition 5.2. *Let $r \in \mathbb{Z}_n$. If r is odd then $\Omega(n, r) \cong \text{Cay}(M(2n, r), S)$. If r is even then $\Omega(n, r) \cong \text{Cay}(M(2n, n - r), S)$.*

Proof. Define a mapping $\phi : V(\text{Cay}(M(2n, r), S)) \rightarrow V(\Omega(n, r))$ by

$$\phi : (x, y) \mapsto (x, y, y).$$

We note that $\text{Cay}(M(2n, r), S)$ and $\text{Cay}(M(2n, n - r), S)$ have the same vertex set so that essentially ϕ is also a mapping from $V(\text{Cay}(M(2n, n - r), S))$ to $V(\Omega(n, r))$. Since n is odd, every y modulo $2n$ maps uniquely to a pair $(y \pmod{2}, y \pmod{n})$ so that ϕ is a bijection.

We note that every edge in $\text{Cay}(M(n, r), S)$ is of the form $[(0, y), (0, y + 1)]$, $[(1, y), (1, y + r)]$ or $[(0, y), (1, y)]$. We split the rest of the proof into two cases.

Case 1: r is odd. We have the following.

$$\begin{aligned}\phi[(0, y), (0, y + 1)] &= [(0, y, y), (0, y + 1, y + 1)] \\ &= [(0, y, y), (0, y, y)(0, 1, \pm 1)]\end{aligned}$$

$$\begin{aligned}\phi[(1, y), (1, y + r)] &= [(1, y, y), (1, y + r, y + r)] \\ &= [(1, y, y), (1, y + 1, y + r)] \\ &= [(1, y, y), (1, y, y)(0, 1, \pm 1)]\end{aligned}$$

$$\begin{aligned}\phi[(0, y), (1, y)] &= [(0, y, y), (1, y, y)] \\ &= [(0, y, y), (0, y, y)(1, 0, 0)]\end{aligned}$$

Thus ϕ is a bijective homomorphism.

Both graphs have the same degree and order of vertex set. Therefore $|E(\text{Cay}(M(2n, r), S))| = |E(\Omega(n, r))|$ and by Lemma 2.5 $\Omega(n, r) \cong \text{Cay}(M(2n, r), S)$ when r is odd.

Case 2: r is even. We have the following.

$$\begin{aligned}\phi[(0, y), (0, y + 1)] &= [(0, y, y), (0, y + 1, y + 1)] \\ &= [(0, y, y), (0, y, y)(0, 1, \pm 1)]\end{aligned}$$

$$\begin{aligned}\phi[(1, y), (1, y + (n - r))] &= [(1, y, y), (1, y + (n - r), y + (n - r))] \\ &= [(1, y, y), (1, y + 1, y - r)] \\ &= [(1, y, y), (1, y, y)(0, 1, \pm 1)]\end{aligned}$$

$$\begin{aligned}\phi[(0, y), (1, y)] &= [(0, y, y), (1, y, y)] \\ &= [(1, y, y), (1, y, y)(1, 0, 0)]\end{aligned}$$

Thus ϕ is a bijective homomorphism.

Both graphs have the same degree and order of vertex set. Therefore $|E(\text{Cay}(M(2n, n - r), S))| = |E(\Omega(n, r))|$ and by Lemma 2.5 $\Omega(n, r) \cong \text{Cay}(M(2n, n - r), S)$ when r is even. \square

Remark 5.3. It is known that the generalised Petersen graph $G(n, r)$ is isomorphic to $G(n, -r)$ [2]. We note that r and $n - r$ have different parity. Thus, in view of Proposition 5.4, we have

$$\Omega(n, r) \cong G(2n, r) \cong G(2n, n - (n - r)) \cong \Omega(n, n - r) \cong \Omega(n, -r)$$

if r is odd, and

$$\Omega(n, r) \cong G(2n, n - r) \cong \Omega(n, n - r) \cong \Omega(n, -r)$$

if r is even.

The automorphism groups of the generalised Petersen graphs were fully determined by Frucht, Graver and Watkins in [4]. In [15] it was shown that they are quasi-Cayley if and only if $r^2 \equiv -1 \pmod{n}$. In [17] it was shown that generalised Petersen graphs are VTNCGs if and only if $r^2 \equiv -1 \pmod{n}$. Therefore we conclude that $\Omega(n, r)$ are VTNCGs.

In view of the above discussion, our consideration of the Cayley set $U_1 = \{(0, 1, 1), (0, 1, -1), (1, 0, 0)\}$ is complete.

5.3 The graphs $\Gamma(n, r)$

We begin by noting the following isomorphic relation. It is similar to the isomorphic relation highlighted in Remark 5.3.

Proposition 5.4. *Let $r \in \mathbb{Z}_n$. Then $\Gamma(n, r) \cong \Gamma(n, -r)$.*

Proof. Define a mapping $\phi : V(\Gamma(n, r)) \longrightarrow V(\Gamma(n, -r))$ by

$$\phi : (x, y, z) \mapsto (x, y, z).$$

We note that the vertex sets are the same and that ϕ maps as an identity map. Therefore it is a bijection.

For any vertex (x, y, z) in V , we have the following.

$$\begin{aligned} \phi([(x, y, z), (x, y, z)(1, 0, 1)]) &= \phi([(x, y, z), (x + 1, y, z + r^x(-1)^y(1))]) \\ &= [(x, y, z), (x + 1, y, z + (-r)^x(-1)^y(\pm 1))] \end{aligned}$$

$$= [(x, y, z), (x, y, z)(1, 0, \pm 1)]$$

$$\begin{aligned} \phi([(x, y, z), (x, y, z)(1, 0, -1)]) &= \phi([(x, y, z), (x + 1, y, z + r^x(-1)^y(-1))]) \\ &= [(x, y, z), (x + 1, y, z + (-r)^x(-1)^y(\pm 1))] \\ &= [(x, y, z), (x, y, z)(1, 0, \pm 1)] \end{aligned}$$

$$\begin{aligned} \phi([(x, y, z), (x, y, z)(1, 0, -r)]) &= \phi([(x, y, z), (x + 1, y, z + r^x(-1)^y(-r))]) \\ &= [(x, y, z), (x + 1, y, z + (-r)^x(-1)^y(\pm r))] \\ &= [(x, y, z), (x, y, z)(1, 0, \pm r)] \end{aligned}$$

$$\begin{aligned} \phi([(x, y, z), (x, y, z)(1, 0, r)]) &= \phi([(x, y, z), (x + 1, y, z + r^x(-1)^y(r))]) \\ &= [(x, y, z), (x + 1, y, z + (-r)^x(-1)^y(\pm r))] \\ &= [(x, y, z), (x, y, z)(1, 0, \pm r)] \end{aligned}$$

$$\begin{aligned} \phi([(x, y, z), (x, y, z)(0, 1, 1)]) &= \phi([(x, y, z), (x, y + 1, z + r^x(-1)^y(1))]) \\ &= [(x, y, z), (x, y + 1, z + (-r)^x(-1)^y(\pm 1))] \\ &= [(x, y, z), (x, y, z)(0, 1, \pm 1)] \end{aligned}$$

$$\begin{aligned} \phi([(x, y, z), (x, y, z)(0, 1, -1)]) &= \phi([(x, y, z), (x, y + 1, z + r^x(-1)^y(-1))]) \\ &= [(x, y, z), (x, y + 1, z + (-r)^x(-1)^y(\pm 1))] \\ &= [(x, y, z), (x, y, z)(0, 1, \pm 1)] \end{aligned}$$

Thus ϕ is a bijective homomorphism.

Both graphs have the same degree and order of vertex set. Therefore $|E(\Gamma(n, r))| = |E(\Gamma(n, -r))|$ and by Lemma 2.5, ϕ is an isomorphism. \square

Our approach to determining the automorphism groups in this case will be similar to the approach taken by Frucht, Graver and Watkins in [4]. To facilitate this, we will partition E . We do this in a way which respects the vertex set partition we defined in the previous chapter. The partition will be as follows.

$$L := \{[u, v] \in E : u \in V_{00}, v \in V_{01}\}; \quad (5.1)$$

$$R := \{[u, v] \in E : u \in V_{10}, v \in V_{11}\}; \quad (5.2)$$

$$M_1 := \{[u, v] \in E : u \in V_{00} \text{ and } v \in V_{10}\}; \quad (5.3)$$

$$M_2 := \{[u, v] \in E : u \in V_{11} \text{ and } v \in V_{01}\}. \quad (5.4)$$

Further to this, we also look at the set

$$M := M_1 \cup M_2, \quad (5.5)$$

so that L, R and M also form a partition of E .

The edge set partitions we have defined will assist in the determination of orbits of certain subsets of the automorphism group. That is, subsets of $\text{Aut } \Gamma(n, r)$ which fix setwise the sets L, R and M . The subsets are $\text{Aut } \Gamma(n, r)_{L,R,M}$ and $\text{Aut } \Gamma(n, r)_M$. Determining the orbits of these particular subsets of $\text{Aut } \Gamma(n, r)$ will in turn lead to the determination of $\text{Aut } \Gamma(n, r)$ itself.

We begin with the following useful lemma.

Lemma 5.5. *Let L, R and M be as defined in (5.1) to (5.5). If $\gamma \in \text{Aut } \Gamma(n, r)$ fixes any of L, R or M set-wise, then it fixes all three or fixes M and interchanges L and R .*

Proof. Suppose γ fixes R set-wise. Any edge in L is incident to four edges in M and one edge in L at either end vertex. Let γ map some edge in L onto some edge in M . Then, of the five edges incident at one end, two have to be mapped to edges in M , since every edge in M is incident to two edges in R . This contradicts that R is fixed. Therefore, if R is fixed, then L and M are also fixed. Similarly, the argument holds if L is fixed.

Suppose γ fixes M set-wise. Now, the edges in L and edges in R form $2n$ -cycles. Thus, if L and R are not fixed or interchanged, then we will have some edge in L being incident to some edge in R , which does not occur. This completes the proof. \square

Our first consideration is finding $\text{Aut } \Gamma(n, r)_{L,R,M}$, the set stabiliser of L, R and M . As mentioned in the proof of Lemma 5.5, the edges of L and R form $2n$ -cycles. Therefore, any automorphism in $\text{Aut } \Gamma(n, r)_{L,R,M}$ is essentially an automorphism of a $2n$ -cycle.

Let Γ be a graph with

$$V(\Gamma) = \{v_0, v_1, \dots, v_{n-1}\} \text{ and } E(\Gamma) = \{[v_i, v_{i+1}] : i \in \mathbb{Z}_n\}.$$

Then Γ consists entirely of just one n -cycle. Define the following maps on $V(\Gamma)$:

$$\rho : v_i \mapsto v_{i+1} \text{ and } \delta : v_i \mapsto v_{-i}.$$

It is easy to show that ρ and δ are automorphisms of Γ . Powers of ρ are essentially rotations on Γ and powers of ρ combined with δ are essentially reflections on Γ . Let $\phi \in \text{Aut } \Gamma$ such that $\phi(v_i) = v_j$. Then we have $\phi(v_{i+1}) = v_{j+1}$ or $\phi(v_{i+1}) = v_{j-1}$. Now, we can write j as $i+k$ for some integer k . If $\phi(v_{i+1}) = v_{i+k+1}$ then $\phi = \rho^k$. If $\phi(v_{i+1}) = v_{i+k-1}$ then $\phi = \rho^{i+k}\delta\rho^{-i}$. Therefore, any automorphism of Γ is contained in $\langle \rho, \delta \rangle$, or we can say that the automorphisms of cycles consist only of rotations and reflections.

In view of the above discussion, we define the following rotations and reflections on the $2n$ -cycle of L :

$$\rho : (x, y, z) \mapsto (x, y+1, z+1); \quad (5.6)$$

$$\delta : (x, y, z) \mapsto (x, y, -z). \quad (5.7)$$

Note that ρ represents a rotation on the $2n$ -cycle of L with $\rho^{2n} = 1$ and δ represents a reflection on L with $\delta^2 = 1$. Further to these mappings respecting the the $2n$ -cycle of L , we also require that they are in fact automorphisms of $\Gamma(n, r)$. In this regard, we present the following lemma.

Lemma 5.6. *Let L, R and M be as defined in (5.1) to (5.5). Let ρ and δ be mappings on $V(\Gamma(n, r))$ defined as in (5.6) and (5.7). Then ρ and δ are automorphisms of $\Gamma(n, r)$. Moreover, $\rho, \delta \in \text{Aut } \Gamma(n, r)_{L,R,M}$.*

Proof. It is easy to see that both ρ and δ are permutations of V .

For any vertex (x, y, z) in V , we have the following

$$\begin{aligned} \rho([(x, y, z), (x, y, z)(1, 0, 1)]) &= \rho([(x, y, z), (x+1, y, z+r^x(-1)^y(1))]) \\ &= [(x, y+1, z+1), (x+1, y+1, z+r^x(-1)^y(1)+1)] \\ &= [(x, y+1, z+1), (x+1, y+1, z+1+r^x(-1)^{y+1}(-1))] \\ &= [(x, y+1, z+1), (x, y+1, z+1)(1, 0, -1)] \end{aligned}$$

$$\begin{aligned} \rho([(x, y, z), (x, y, z)(1, 0, -1)]) &= \rho([(x, y, z), (x+1, y, z+r^x(-1)^y(-1))]) \\ &= [(x, y+1, z+1), (x+1, y+1, z+r^x(-1)^y(-1)+1)] \\ &= [(x, y+1, z+1), (x+1, y+1, z+1+r^x(-1)^{y+1}(1))] \end{aligned}$$

$$= [(x, y + 1, z + 1), (x, y + 1, z + 1)(1, 0, 1)]$$

$$\begin{aligned} \rho([(x, y, z), (x, y, z)(1, 0, r)]) &= \rho([(x, y, z), (x + 1, y, z + r^x(-1)^y(r))]) \\ &= [(x, y + 1, z + 1), (x + 1, y + 1, z + r^x(-1)^y(r) + 1)] \\ &= [(x, y + 1, z + 1), (x + 1, y + 1, z + 1 + r^x(-1)^{y+1}(-r))] \\ &= [(x, y + 1, z + 1), (x, y + 1, z + 1)(1, 0, -r)] \end{aligned}$$

$$\begin{aligned} \rho([(x, y, z), (x, y, z)(1, 0, -r)]) &= \rho([(x, y, z), (x + 1, y, z + r^x(-1)^y(-r))]) \\ &= [(x, y + 1, z + 1), (x + 1, y + 1, z + r^x(-1)^y(-r) + 1)] \\ &= [(x, y + 1, z + 1), (x + 1, y + 1, z + 1 + r^x(-1)^{y+1}(r))] \\ &= [(x, y + 1, z + 1), (x, y + 1, z + 1)(1, 0, r)] \end{aligned}$$

$$\begin{aligned} \rho([(x, y, z), (x, y, z)(0, 1, 1)]) &= \rho([(x, y, z), (x, y + 1, z + r^x(-1)^y(1))]) \\ &= [(x, y + 1, z + 1), (x, y, z + r^x(-1)^y(1) + 1)] \\ &= [(x, y + 1, z + 1), (x, y, z + 1 + r^x(-1)^{y+1}(-1))] \\ &= [(x, y + 1, z + 1), (x, y + 1, z + 1)(0, 1, -1)] \end{aligned}$$

$$\begin{aligned} \rho([(x, y, z), (x, y, z)(0, 1, -1)]) &= \rho([(x, y, z), (x, y + 1, z + r^x(-1)^y(-1))]) \\ &= [(x, y + 1, z + 1), (x, y, z + r^x(-1)^y(-1) + 1)] \\ &= [(x, y + 1, z + 1), (x, y, z + 1 + r^x(-1)^{y+1}(1))] \\ &= [(x, y + 1, z + 1), (x, y + 1, z + 1)(0, 1, 1)] \end{aligned}$$

so that ρ preserves edges. We also have

$$\begin{aligned} \delta([(x, y, z), (x, y, z)(1, 0, 1)]) &= \delta([(x, y, z), (x + 1, y, z + r^x(-1)^y(1))]) \\ &= [(x, y, -z), (x + 1, y, -z - r^x(-1)^y(1))] \\ &= [(x, y, -z), (x + 1, y, -z + r^x(-1)^y(-1))] \\ &= [(x, y, -z), (x, y, -z)(1, 0, -1)] \end{aligned}$$

$$\begin{aligned} \delta([(x, y, z), (x, y, z)(1, 0, -1)]) &= \delta([(x, y, z), (x + 1, y, z + r^x(-1)^y(-1))]) \\ &= [(x, y, -z), (x + 1, y, -z - r^x(-1)^y(-1))] \\ &= [(x, y, -z), (x + 1, y, -z + r^x(-1)^y(1))] \\ &= [(x, y, -z), (x, y, -z)(1, 0, 1)] \end{aligned}$$

$$\begin{aligned}
\delta([(x, y, z), (x, y, z)(1, 0, r)]) &= \delta([(x, y, z), (x + 1, y, z + r^x(-1)^y(r))]) \\
&= [(x, y, -z), (x + 1, y, -z - r^x(-1)^y(r))] \\
&= [(x, y, -z), (x + 1, y, -z + r^x(-1)^y(-r))] \\
&= [(x, y, -z), (x, y, -z)(1, 0, -r)]
\end{aligned}$$

$$\begin{aligned}
\delta([(x, y, z), (x, y, z)(1, 0, -r)]) &= \delta([(x, y, z), (x + 1, y, z + r^x(-1)^y(-r))]) \\
&= [(x, y, -z), (x + 1, y, -z - r^x(-1)^y(-r))] \\
&= [(x, y, -z), (x + 1, y, -z + r^x(-1)^y(r))] \\
&= [(x, y, -z), (x, y, -z)(1, 0, r)]
\end{aligned}$$

$$\begin{aligned}
\delta([(x, y, z), (x, y, z)(0, 1, 1)]) &= \delta([(x, y, z), (x, y + 1, z + r^x(-1)^y(1))]) \\
&= [(x, y, -z), (x, y + 1, -z - r^x(-1)^y(1))] \\
&= [(x, y, -z), (x, y + 1, -z + r^x(-1)^y(-1))] \\
&= [(x, y, -z), (x, y, -z)(0, 1, -1)]
\end{aligned}$$

$$\begin{aligned}
\delta([(x, y, z), (x, y, z)(0, 1, -1)]) &= \delta([(x, y, z), (x, y + 1, z + r^x(-1)^y(-1))]) \\
&= [(x, y, -z), (x, y + 1, -z - r^x(-1)^y(-1))] \\
&= [(x, y, -z), (x, y + 1, -z + r^x(-1)^y(1))] \\
&= [(x, y, -z), (x, y, -z)(0, 1, -1)]
\end{aligned}$$

so that δ preserves edges. Therefore, ρ and δ are automorphisms of $\Gamma(n, r)$.

Moreover, we can see that ρ maps vertices in V_{00} to vertices in V_{01} and vice versa so that L is fixed setwise. By Lemma 5.5 R and M are also fixed setwise. Also, δ fixes each set in our partition of V setwise so that naturally, R, L and M are fixed. \square

As it turns out, $\text{Aut } \Gamma(n, r)_{L, R, M}$ can be described entirely by ρ and δ . The following lemma proves this.

Lemma 5.7. *Let L, R and M be as defined in (5.1) to (5.5). Let $\text{Aut } \Gamma(n, r)_{L, R, M}$ be the set stabiliser of L, R, M in $\text{Aut } \Gamma(n, r)$, and ρ, δ be the maps defined as in (5.6) and (5.7). Then $\text{Aut } \Gamma(n, r)_{L, R, M} = \langle \rho, \delta \rangle$.*

Proof. Any automorphism in $\text{Aut } \Gamma(n, r)_{L, R, M}$ must preserve the $2n$ -cycles induced

by the edges in L and R . All the automorphisms which preserve the induced $2n$ -cycles are contained in $\langle \rho, \delta \rangle$. Therefore $\text{Aut } \Gamma(n, r)_{L, R, M} = \langle \rho, \delta \rangle$. \square

Now that we have determined $\text{Aut } \Gamma(n, r)_{L, R, M}$ completely, let us now consider automorphisms in $\text{Aut } \Gamma(n, r)_M$ which interchange L and R . By Lemma 5.5 we will then have entirely considered $\text{Aut } \Gamma(n, r)_M$.

We define another mapping as follows.

$$\alpha : (x, y, z) \mapsto (x + 1, y, rz). \quad (5.8)$$

As we did with ρ and δ , we want to confirm that α is an automorphism of $\Gamma(n, r)$.

Lemma 5.8. *Let α be a mapping on $V(\Gamma(n, r))$ as defined in (5.8). Then α is an automorphism of $\Gamma(n, r)$. Moreover, $\alpha \in \text{Aut } \Gamma(n, r)_M$.*

Proof. It is easy to see that α is a permutation of V since $(n, r) = 1$.

Recall that the exponent of r is modulo 2 so that rr^x is not necessarily equal to r^{x+1} . For any vertex of the form $(0, y, z)$ we have the following

$$\begin{aligned} \alpha([(0, y, z), (0, y, z)(1, 0, 1)]) &= \alpha([(0, y, z), (1, y, z + r^0(-1)^y(1))]) \\ &= [(1, y, rz), (0, y, rz + r(-1)^y(1))] \\ &= [(1, y, rz), (1, y, rz)(1, 0, 1)] \end{aligned}$$

$$\begin{aligned} \alpha([(0, y, z), (0, y, z)(1, 0, -1)]) &= \alpha([(0, y, z), (1, y, z + r^0(-1)^y(-1))]) \\ &= [(1, y, rz), (0, y, rz + r(-1)^y(-1))] \\ &= [(1, y, rz), (1, y, rz)(1, 0, -1)] \end{aligned}$$

$$\begin{aligned} \alpha([(0, y, z), (0, y, z)(1, 0, r)]) &= \alpha([(0, y, z), (1, y, z + r^0(-1)^y(r))]) \\ &= [(1, y, rz), (0, y, rz + r(-1)^y(r))] \\ &= [(1, y, rz), (1, y, rz)(1, 0, r)] \end{aligned}$$

$$\begin{aligned} \alpha([(0, y, z), (0, y, z)(1, 0, -r)]) &= \alpha([(0, y, z), (1, y, z + r^0(-1)^y(-r))]) \\ &= [(1, y, rz), (0, y, rz + r(-1)^y(-r))] \\ &= [(1, y, rz), (1, y, rz)(1, 0, -r)] \end{aligned}$$

$$\begin{aligned}
\alpha([(0, y, z), (0, y, z)(0, 1, 1)]) &= \alpha([(0, y, z), (0, y + 1, z + r^0(-1)^y(1))]) \\
&= [(1, y, rz), (1, y + 1, rz + r(-1)^y(1))] \\
&= [(1, y, rz), (1, y, rz)(0, 1, 1)]
\end{aligned}$$

$$\begin{aligned}
\alpha([(0, y, z), (0, y, z)(0, 1, -1)]) &= \alpha([(0, y, z), (0, y + 1, z + r^0(-1)^y(-1))]) \\
&= [(1, y, rz), (1, y + 1, rz + r(-1)^y(-1))] \\
&= [(1, y, rz), (1, y, rz)(0, 1, -1)]
\end{aligned}$$

For any vertex of the form $(1, y, z)$ we have the following

$$\begin{aligned}
\alpha([(1, y, z), (1, y, z)(1, 0, 1)]) &= \alpha([(1, y, z), (0, y, z + r(-1)^y(1))]) \\
&= [(0, y, rz), (1, y, rz + rr(-1)^y(1))] \\
&= [(0, y, rz), (1, y, rz + r^0(-1)^y(-1))] \\
&= [(0, y, rz), (0, y, rz)(1, 0, -1)]
\end{aligned}$$

$$\begin{aligned}
\alpha([(1, y, z), (1, y, z)(1, 0, -1)]) &= \alpha([(1, y, z), (0, y, z + r(-1)^y(-1))]) \\
&= [(0, y, rz), (1, y, rz + rr(-1)^y(-1))] \\
&= [(0, y, rz), (1, y, rz + r^0(-1)^y(1))] \\
&= [(0, y, rz), (0, y, rz)(1, 0, 1)]
\end{aligned}$$

$$\begin{aligned}
\alpha([(1, y, z), (1, y, z)(1, 0, r)]) &= \alpha([(1, y, z), (0, y, z + r(-1)^y(r))]) \\
&= [(0, y, rz), (1, y, rz + rr(-1)^y(r))] \\
&= [(0, y, rz), (1, y, rz + r^0(-1)^y(-r))] \\
&= [(0, y, rz), (0, y, rz)(1, 0, -r)]
\end{aligned}$$

$$\begin{aligned}
\alpha([(1, y, z), (1, y, z)(1, 0, -r)]) &= \alpha([(1, y, z), (0, y, z + r(-1)^y(-r))]) \\
&= [(0, y, rz), (1, y, rz + rr(-1)^y(-r))] \\
&= [(0, y, rz), (1, y, rz + r^0(-1)^y(r))] \\
&= [(0, y, rz), (0, y, rz)(1, 0, r)]
\end{aligned}$$

$$\begin{aligned}
\alpha([(1, y, z), (1, y, z)(0, 1, 1)]) &= \alpha([(1, y, z), (1, y + 1, z + r(-1)^y(1))]) \\
&= [(0, y, rz), (0, y + 1, rz + rr(-1)^y(1))]
\end{aligned}$$

$$\begin{aligned}
&= [(0, y, rz), (0, y + 1, rz + r^0(-1)^y(-1))] \\
&= [(0, y, rz), (1, y, rz)(0, 1, -1)] \\
\alpha([(1, y, z), (1, y, z)(0, 1, -1)]) &= \alpha([(1, y, z), (1, y + 1, z + r(-1)^y(-1))]) \\
&= [(0, y, rz), (0, y + 1, rz + rr(-1)^y(-1))] \\
&= [(0, y, rz), (0, y + 1, rz + r^0(-1)^y(1))] \\
&= [(0, y, rz), (1, y, rz)(0, 1, 1)]
\end{aligned}$$

Therefore, α preserves edges.

Moreover, we can see that α maps vertices in V_{00} to vertices in V_{10} and vertices in V_{01} to vertices in V_{11} . Thus, all edges in L are mapped to edges in R and vice versa. Therefore, α fixes M setwise. \square

Remark 5.9. We note that

$$\begin{aligned}
\alpha^2(x, y, z) &= \alpha(x + 1, y, rz) \\
&= (x, y, -z)
\end{aligned}$$

and

$$\begin{aligned}
\alpha^4(x, y, z) &= \alpha^3(x + 1, y, rz) \\
&= \alpha^2(x, y, -rz) \\
&= \alpha(x + 1, y, -rz) \\
&= (x, y, z)
\end{aligned}$$

so that we have $\alpha^2 = \delta$ and $\alpha^4 = 1$.

We aim now to determine $\text{Aut } \Gamma(n, r)_M$ completely. First, we will need the following lemma, after which we proceed to determine $\text{Aut } \Gamma(n, r)_M$.

Lemma 5.10. *Let $\gamma \in \text{Aut } \Gamma(n, r)_M$ interchange L and R . If $\gamma(0, y, z) = (1, y', z')$ then $\gamma(1, y, z) = (0, y', z')$.*

Proof. If $\gamma(0, y, z) = (1, y', z')$ then $\gamma(0, y + 1, z + 1) = (1, y' + 1, z' + r)$ and $\gamma(0, y + 1, z - 1) = (1, y' + 1, z' - r)$, or $\gamma(0, y + 1, z + 1) = (1, y' + 1, z' - r)$ and $\gamma(0, y + 1, z - 1) = (1, y' + 1, z' + r)$. Consequently, we have that $\gamma(0, y + i, z + i) = (1, y' + i, z' + ir)$ or $\gamma(0, y + i, z + i) = (1, y' + i, z' - ir)$. In which case, the 4-cycle

$$(0, y + 1, z)(0, y, z + 1)(1, y, z)(0, y, z - 1)$$

is mapped to

$$(1, y' + 1, z')(1, y', z' \pm r)\gamma(1, y, z)(1, y', z' \pm r).$$

Therefore, we must have $\gamma(1, y, z) = (0, y', z')$. \square

Lemma 5.11. *Let ρ, δ and α be as in (5.6), (5.7) and (5.8). Then $\text{Aut } \Gamma(n, r)_M = \langle \rho, \alpha \rangle$.*

Proof. Let $\gamma \in \text{Aut } \Gamma(n, r)_M$ interchange L and R . By Lemma 5.10, we can compose γ with some power of ρ such that $(1, 0, 0)$ and $(0, 0, 0)$ are interchanged. We may also compose with δ such that $(0, 1, 1)$ is mapped to $(1, 1, r)$. It then follows that $(0, i, i)$ is mapped to $(1, i, ir)$ and $(1, i, i)$ is mapped to $(0, i, ir)$. Thus, we have that $\gamma\rho^k\delta^h = \alpha$ for some integers k and h . That is to say, any automorphism in $\text{Aut } \Gamma(n, r)_M$ can be written as a combination of powers of ρ, δ and α . Now recall that $\alpha^2 = \delta$. Therefore, we have that $\text{Aut } \Gamma(n, r)_M = \langle \rho, \alpha \rangle$. \square

Our aim is to completely determine $\text{Aut } \Gamma(n, r)$. Thus far, we have determined all automorphisms of $\Gamma(n, r)$ which fix M setwise. Of course there may exist automorphisms which do not fix M . These are our next consideration. We need to identify these automorphisms if they exist. For that, we will need the following lemma.

Lemma 5.12. *$\Gamma(n, r)$ is edge-transitive if and only if $\text{Aut } \Gamma(n, r)_M$ is a proper subgroup of $\text{Aut } \Gamma(n, r)$.*

Proof. If $\Gamma(n, r)$ is edge-transitive then there exist an automorphism in $\text{Aut } \Gamma(n, r)$ which maps an edge in M to an edge L . It follows immediately that $\text{Aut } \Gamma(n, r)_M$ is a proper subgroup of $\text{Aut } \Gamma(n, r)$.

If $\text{Aut } \Gamma(n, r)$ is not edge-transitive then it has at least two edge orbits. Also, it is impossible for $\text{Aut } \Gamma(n, r)$ to contain more edge orbits than $\text{Aut } \Gamma(n, r)_M$. Since $L \cup R$ and M form the only two edge orbits in $\text{Aut } \Gamma(n, r)_M$, they must be the only two edge orbits in $\text{Aut } \Gamma(n, r)$. Therefore, $\text{Aut } \Gamma(n, r)$ coincides with $\text{Aut } \Gamma(n, r)_M$, contradicting the fact that $\text{Aut } \Gamma(n, r)_M$ is a proper subgroup of $\text{Aut } \Gamma(n, r)$. \square

In view of Lemma 5.12, we naturally aim to find those cases of $\Gamma(n, r)$ which are not edge-transitive. If it happens that there exists no edge-transitive graphs in our class of graphs, then our determination of the automorphism group is complete. To

facilitate our search for edge-transitive graphs we will define certain parameters as was done in [4].

Let C be a cycle in $\Gamma(n, r)$. We define the following:

$$\begin{aligned} l(C) &:= \text{number of edges in } C \cap L; \\ r(C) &:= \text{number of edges in } C \cap R; \\ m(C) &:= \text{number of edges in } C \cap M. \end{aligned}$$

Let Δ be the set of all 4-cycles in $\Gamma(n, r)$, and further define

$$L_4 := \sum_{C \in \Delta} l(C); \quad (5.9)$$

$$R_4 := \sum_{C \in \Delta} r(C); \quad (5.10)$$

$$M_4 := \sum_{C \in \Delta} m(C). \quad (5.11)$$

The quantities L_4, R_4 and M_4 effectively count the number of times any edge in L, R and M appears in any 4-cycle.

The following lemma relates the quantities L_4, R_4 and M_4 to whether $\text{Aut } \Gamma(n, r)_M$ is a proper subgroup of $\text{Aut } \Gamma(n, r)$ or not. By Lemma 5.12, we are then able to determine edge-transitivity of our graphs by calculating the quantities L_4, R_4 and M_4 .

Lemma 5.13. *Let L_4, R_4 and M_4 be as in (5.9), (5.10) and (5.11). If $\text{Aut } \Gamma(n, r)_M \neq \text{Aut } \Gamma(n, r)$ then $4L_4 = 4R_4 = M_4$.*

Proof. Assume that $\text{Aut } \Gamma(n, r)_M \neq \text{Aut } \Gamma(n, r)$. By Lemma 5.12, $\Gamma(n, r)$ is edge-transitive. Let e be an edge involved in k 4-cycles. By edge-transitivity, every edge is involved in k 4-cycles. Since we have $2n$ edges in L , $2n$ edges in R and $8n$ edges in M , we have that $L_4 = 2nk$, $R_4 = 2nk$ and $M_4 = 8nk$. Therefore, $4L_4 = 4R_4 = M_4$. \square

In view of Lemma 5.13, we will now count the number of 4-cycles in $\Gamma(n, r)$. Using the lemmas we have proved, we may then determine whether $\text{Aut } \Gamma(n, r)_M$ is a proper subgroup of $\text{Aut } \Gamma(n, r)$ or not.

We distinguish between different types of 4-cycles for ease of counting. We therefore partition Δ , the set of all 4-cycles in $\Gamma(n, r)$, as follows:

$$A_1 := \{C \in \Delta : |C \cap M_1| = 4\}; \quad (5.12)$$

$$A_2 := \{C \in \Delta : |C \cap M_2| = 4\}; \quad (5.13)$$

$$B_1 := \{C \in \Delta : |C \cap M_1| = 2 \text{ and } |C \cap R| = 2\}; \quad (5.14)$$

$$B_2 := \{C \in \Delta : |C \cap M_2| = 2 \text{ and } |C \cap R| = 2\}; \quad (5.15)$$

$$D_1 := \{C \in \Delta : |C \cap M_1| = 2 \text{ and } |C \cap L| = 2\}; \quad (5.16)$$

$$D_2 := \{C \in \Delta : |C \cap M_2| = 2 \text{ and } |C \cap L| = 2\}; \quad (5.17)$$

$$E := \{C \in \Delta : |C \cap M| = 2 \text{ and } |C \cap L| = 1 \text{ and } |C \cap R| = 1\}. \quad (5.18)$$

We also define the sets A, B and D as:

$$A := A_1 \cup A_2; \quad (5.19)$$

$$B := B_1 \cup B_2; \quad (5.20)$$

$$D := D_1 \cup D_2, \quad (5.21)$$

so that A, B, D and E form a partition of Δ .

The following lemma counts the number of 4-cycles in $\Gamma(n, r)$ and calculates the quantities L_4, R_4 and M_4 for each n .

Lemma 5.14. *Let L_4, R_4 and M_4 be as in (5.9), (5.10) and (5.11). Then $4R_4 = 4L_4 \neq M_4$ in $\Gamma(n, r)$ for any n .*

Proof. Let $A_1, A_2, B_1, B_2, D_1, D_2, E, A, B$ and D be as in (5.12) to (5.21). The vertex $(0, 0, z)$ in V_{00} is involved in the following four 4-cycles in A_1 :

- 1) $(0, 0, z)(1, 0, z - r)(0, 0, z - r - 1)(1, 0, z - 1)$;
- 2) $(0, 0, z)(1, 0, z - 1)(0, 0, z - 1 + r)(1, 0, z + r)$;
- 3) $(0, 0, z)(1, 0, z + 1)(0, 0, z + 1 - r)(1, 0, z - r)$;
- 4) $(0, 0, z)(1, 0, z + r)(0, 0, z + r + 1)(1, 0, z + 1)$.

The second and fourth vertices in the above 4-cycles cover all four of the vertices adjacent to $(0, 0, z)$ in V_{10} . Only if the third co-ordinate in any of the third vertices above equals an element in $\{z - 2, z - 2r, z + 2, z + 2r\}$ does there exist another

4-cycle in A_1 involving $(0, 0, z)$. We solve the following equations to find possible instances where such exceptions exist. By considering parities, it is only these eight equations that we need to consider:

- (i) $2 = -r - 1 \Rightarrow -r = 3 \Rightarrow -1 = 9 \Rightarrow n = 5$;
- (ii) $2 = r - 1 \Rightarrow r = 3 \Rightarrow -1 = 9 \Rightarrow n = 5$;
- (iii) $2 = -r + 1 \Rightarrow r = -1$, which is impossible;
- (iv) $2 = r + 1 \Rightarrow r = 1$, which is impossible;
- (v) $2r = -r - 1 \Rightarrow 3r = 1 \Rightarrow -9 = 1 \Rightarrow n = 5$;
- (vi) $2r = r - 1 \Rightarrow r = -1$, which is impossible;
- (vii) $2r = -r + 1 \Rightarrow 3r = 1 \Rightarrow -9 = 1 \Rightarrow n = 5$;
- (viii) $2r = r + 1 \Rightarrow r = 1$, which is impossible.

Thus the only exceptional case is when $n = 5$. For now, we consider only $n > 5$.

If each vertex in V_{00} is involved in four 4-cycles in A_1 then we have $4n$ 4-cycles in A_1 . However, in the $4n$ we would have counted each cycle twice since each $(0, 0, z)$ will appear again as the third vertex in the consideration of some other $(0, 0, z')$. Thus, we have $2n$ 4-cycles in A_1 . It is clear that the order of A_1 equals the order of A_2 , so that the order of A is $4n$.

$(0, 0, z)$ is involved in only one cycle in B_1 :

- 1) $(0, 0, z)(1, 0, z + r)(1, 1, z)(1, 0, z - r)$.

so that the order of B_1 is n . It is easy to see that the order of B_1 equals the order of B_2 , so that the order of B is $2n$. Similarly, the order of D is $2n$.

$(0, 0, z)$ is involved in eight 4-cycles in E :

- 1) $(0, 0, z)(0, 1, z - 1)(1, 1, z)(1, 0, z + r)$;
- 2) $(0, 0, z)(0, 1, z - 1)(1, 1, z)(1, 0, z - r)$;

- 3) $(0, 0, z)(0, 1, z - 1)(1, 1, z - 1 + r)(1, 0, z - 1)$;
- 4) $(0, 0, z)(0, 1, z - 1)(1, 1, z - 1 - r)(1, 0, z - 1)$;
- 5) $(0, 0, z)(0, 1, z + 1)(1, 1, z + 1 + r)(1, 0, z + 1)$;
- 6) $(0, 0, z)(0, 1, z + 1)(1, 1, z + 1 - r)(1, 0, z + 1)$;
- 7) $(0, 0, z)(0, 1, z + 1)(1, 1, z)(1, 0, z + r)$;
- 8) $(0, 0, z)(0, 1, z + 1)(1, 1, z)(1, 0, z - r)$.

Similar to cycles in A , we determine possible exceptional cases. In this case, an exception will occur if an element of $\{z - 2, z - 2r, z + 2, z + 2r\}$ equals an element of $\{z - r - 1, z - 1 + r, z + 1 - r, z + r + 1, z\}$. This is the same as before except that we now need to consider when an element of $\{z - 2, z - 2r, z + 2, z + 2r\}$ equals z . Clearly, this is impossible. Thus we have that the order of E is $8n$.

Table 5.1 is a presentation of L_4 , R_4 and M_4 for A, B, D and E in $\Gamma(n, r)$ when $n > 5$.

Set	4-cycles	L_4	R_4	M_4
A	$4n$	0	0	$16n$
B	$2n$	0	$4n$	$4n$
D	$2n$	$4n$	0	$4n$
E	$8n$	$8n$	$8n$	$16n$
All	$16n$	$12n$	$12n$	$40n$

Table 5.1: Edge-counts in $\Gamma(n, r)$ for $n > 5$.

Let us now consider the case $n = 5$. If $n = 5$ then we can have $r = 2$ or $r = 3$. However, by Proposition 5.4 we have that $\Gamma(5, 2) \cong \Gamma(5, 3)$. Thus we need only consider when $r = 2$.

We observe that each $|V_{xy}| = 5$. We count the 4-cycles in A . Each vertex (x, y, i) is adjacent to four out of five vertices in $V_{(x+1)y}$. Without loss of generality, we consider the vertex $(0, 0, 0)$ and count the 4-cycles in A which $(0, 0, 0)$ is involved in. Starting at $(0, 0, 0)$ we have four choices for the second vertex $(1, 0, z_1)$. We then have three choices for the third vertex $(0, 0, z_2)$ since we cannot trace back to $(0, 0, 0)$ or $(0, 0, z_2)$. For the fourth vertex we cannot trace back to $(1, 0, 0)$, $(1, 0, z_1)$ or $(1, 0, z_2)$.

so that we have two choices for the fourth vertex $(1, 0, z_3)$. Therefore, taking double counting into account, $(0, 0, 0)$ is involved in $\frac{4 \times 3 \times 2}{2} = 12$ type A 4-cycles. The order of A is then $12 \times 10 = 120$.

We now count 4-cycles in B and D . Again we consider $(0, 0, 0)$. There are exactly three vertices in V_{11} which have in common with $(0, 0, 0)$ two incident vertices in V_{01} . Similarly, the same applies for V_{10} . Therefore, $(0, 0, 0)$ is involved in three 4-cycles in B and three in D . By similar arguments, the order of B and D equals $\frac{20 \times 3}{2} = 30$.

Considering now type E 4-cycles. There exist two paths from $(0, 0, 0)$ to $(1, 1, 0)$ via V_{10} and two via V_{01} , so that we have four 4-cycles in E involving both $(0, 0, 0)$ and $(1, 1, 0)$. Similarly, we have two 4-cycles in E involving $(0, 0, 0)$ and $(1, 1, 1)$, two involving $(0, 0, 0)$ and $(1, 1, 2)$, two involving $(0, 0, 0)$ and $(1, 1, 3)$, and two involving $(0, 0, 0)$ and $(1, 1, 4)$. In total then, we have 12 involving $(0, 0, 0)$, so that we have $12 \times 5 = 60$ type E 4-cycles in total. We summarise in Table 5.2.

Set	4-cycles	L_4	R_4	M_4
A	120	0	0	480
B	30	0	60	60
D	30	60	0	60
E	60	60	60	120
All	240	120	120	720

Table 5.2: Edge-counts in $\Gamma(5, 2)$.

By table 5.1 we can see that $4L_4 = 4R_4 \neq M_4$ for any $n \neq 5$. By table 5.2 we can see that $4L_4 = 4R_4 \neq M_4$ for $n = 5$. Therefore $4L_4 = 4R_4 \neq M_4$ for any n . \square

The following theorem completes our consideration of the automorphism groups.

Theorem 5.15. *Let ρ and α be as in (5.6) and (5.8). Then $\text{Aut } \Gamma(n, r) = \langle \rho, \alpha \rangle$ for any n .*

Proof. By Lemma 5.14 we have that $4L_4 = 4R_4 \neq M_4$ in $\Gamma(n, r)$ for any n . By Lemma 5.13 we have that $\text{Aut } \Gamma(n, r)_M = \text{Aut } \Gamma(n, r)$. By Lemma 5.11 $\text{Aut } \Gamma(n, r) = \langle \rho, \alpha \rangle$. \square

Having completely determined the automorphism groups of $\Gamma(n, r)$, applying Sabidussi's theorem [19] proves to be a relatively simple matter.

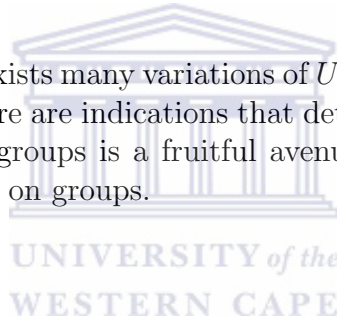
Theorem 5.16. $\Gamma(n, r)$ is a VTNCG.

Proof. It suffices to show non-Cayleyness on groups. By Theorem 5.1, we require a subgroup of $\text{Aut } \Gamma(n, r)$ of order $4n$ which acts transitively on $V(\Gamma(n, r))$ for Cayleyness on groups.

Since ρ fixes $V_{00} \cup V_{01}$ and $V_{10} \cup V_{11}$, $\langle \rho \rangle$ is not transitive. Thus, it is clear that any transitive subgroup of $\text{Aut } \Gamma(n, r) = \langle \rho, \alpha \rangle$ must contain a subgroup of $\langle \alpha \rangle$. Since $\alpha^4 = 1$ and α^2 fixes $V_{00} \cup V_{01}$ and $V_{10} \cup V_{11}$ as well, the only subgroup which suffices is $\langle \alpha \rangle$ itself.

Now, $|\langle \alpha \rangle| = 4$ and the required subgroup must have order $4n$. Also, no power of ρ is equal to any power of α except that $\alpha^4 = \rho^{2n} = 1$. We must therefore include a subgroup of $\langle \rho \rangle$ of order n . The only possibility is $\langle \rho^2 \rangle$. In which case the vertex $(0, 0, z)$ cannot be mapped to any vertex $(0, 1, z')$. Therefore, there exists no subgroup of $\text{Aut } \Gamma(n, r)$ which acts regularly on $V(\Gamma(n, r))$. \square

It is expected that there exists many variations of U such that $\text{Cay}(Q(n, r), U)$ is a VTNCG. More generally, there are indications that determining meta-Cayley graphs on a semi-direct product of groups is a fruitful avenue in finding vertex transitive graphs which are non-Cayley on groups.



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