On the Method of Lines for Singularly Perturbed Partial Differential Equations

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A thesis submitted in partial fulfilment of the requirements for degree of Master of Science in the Department of Mathematics and Applied Mathematics, University of the Western Cape.

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Keywords

Singularly perturbed problems

Partial differential equation

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Boundary layer

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Shishkin mesh

Finite mesh finite difference methods

Fitted operator finite difference methods

Error analysis

Abstract

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Many chemical and physical problems are mathematically described by partial differential equations (PDEs). These PDEs are often highly nonlinear and therefore have no closed form solutions. Thus, it is necessary to recourse to numerical approaches to determine suitable approximations to the solution of such equations. For solutions possessing sharp spatial transitions (such as boundary or interior layers), standard numerical methods have shown limitations as they fail to capture large gradients. The method of lines (MOL) is one of the numerical methods used to solve PDEs. It proceeds by the discretization of all but one dimension leading to systems of ordinary differential equations. In the case of time-dependent PDEs, the MOL consists of discretizing the spatial derivatives only leaving the time variable continuous. The process results in a system to which a numerical method for initial value problems can be applied. In this project we consider various types of singularly perturbed time-dependent PDEs. For each type, using the MOL, the spatial dimensions will be discretized in many different ways following fitted numerical approaches. Each discretisation will be analysed for stability and convergence. Extensive experiments will be conducted to confirm the analyses.

September 2017.

Declaration

I, the undersigned, hereby declare that **On the method of lines for singularly perturbed Partial Differential Equations** is my original work, it has not been submitted before for any degree or examination in any other university, and that any work done by others or myself previously have been acknowledged and referenced accordingly.

Nana Adjoah Mbroh September 2017

Signed:..

Dedication

To my mum and dad.

Acknowledgement

First and foremost, I thank the Almighty God for giving me the strength to be able to withstand all odds to write this dissertation successfully. My deepest gratitude goes to my supervisor, Dr. Justin B. Munyakazi, for your patience, guidance and constructive criticisms, encouragement and above all the time you spent with me to make this work a success.

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To my parents Isaac and Comfort, my sisters Linda and Irene, my beloved brother Eric together with my nieces, I say God bless you all for your prayers, constant encouragement and resources. I would never have wished for a family better than what I have.

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Chapter 1

Introduction

In this chapter, we give a general description of singularly perturbed problems and their numerical treatments. In line with the objective of this thesis, we introduce the fitted finite difference methods and the method of lines. Subsequently, we review some related literature and also present the summary of this thesis in the last section of this chapter.

1.1 Singularly perturbed problems

Many practical problems arising from the development of science and technology are described by parameter dependent differential equations. These equations underly perturbation problems. There are two categories of perturbation problems: regular and singular perturbation problems. A problem P_ε is called regular if the smoothness of its solution $u(x,\varepsilon)$ depends on a parameter $0 < \varepsilon << 1$. Else, P_ε is a singular perturbation problem (SPP). In SPPs, the parameter ε, called the singular perturbation parameter, multiplies the highest derivative term of the differential equation underlying the problem P_ε. As a consequence, if one sets $\varepsilon = 0$, the order of the equation drops. This is not the case for regular perturbation problems.

To be more precise, setting $\varepsilon = 0$, we obtain a reduced problem which we denote by P_0 whose solution we denote by $u(x, 0)$. If

$$\lim_{\varepsilon \to 0} u(x,\varepsilon) = u(x,0),$$

then P_ε is a regular perturbation problem, otherwise it is an SPP.

Solutions of SPPs typically contain layers. For the purpose of this research we consider only linear time-dependent problems. To illustrate the layer behaviour, we follow the works of [34, 55] and present some examples.

Example 1.1.1. *Consider the reaction-diffusion problem*

$$\mathcal{L}u \equiv u_t - \varepsilon u_{xx} + b(x,t)u = f, \ (x,t) \in Q = \Omega \times (0,T], \ \Omega = (0,1), \tag{1.1.1}$$

$$\textit{with the conditions } u(0,t) = \eta_0, \ u(1,t) = \eta_1 \textit{ and } u(x,0) = \varphi(x). \tag{1.1.2}$$

Here and in the rest of this work, $u_t \equiv \partial u/\partial t$, $u_{xx} \equiv \partial^2 u/\partial x^2$. Setting $\varepsilon = 0$ in (1.1.1) gives the initial value problem $u_t + b(x,t)u = f$, along with the conditions (1.1.2). Clearly, when solving the reduced problem, we require none of the boundary conditions, thus the solution will exhibit two boundary layers in the respective boundaries of the spatial domain. Figure 1.1 illustrates the two boundary layers occurring in the solution of problem (1.1.1)-(1.1.2) at a fixed time.

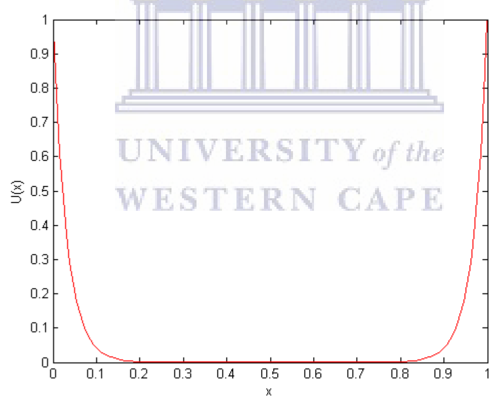

Figure 1.1: Solution of Example 1.1.1 displaying the two boundary layers at a prescribed time.

Example 1.1.2. *Consider the convection-diffusion problem*

$$\mathcal{L}u \equiv u_t - \varepsilon u_{xx} + a(x,t)u_x + b(x,t)u = f, \ (x,t) \in Q = \Omega \times (0,T], \ \Omega = (0,1), \tag{1.1.3}$$

$$\textit{with the conditions}, \ u(0,t) = \eta_0, \ u(1,t) = \eta_1, \textit{ and } u(x,0) = \varphi(x). \tag{1.1.4}$$

Here, setting $\varepsilon = 0$ will result in the first order equation $u_t + a(x,t)u_x + b(x,t) = 0$, along with the conditions (1.1.4). It turns out that only one boundary condition and the initial condition are required for the determination of the analytic solution. Thus a boundary layer will occur at $x = 1$. Note that for a negative convective term, the layer will occur at the neighbourhood of $x = 0$. We illustrate the boundary layer in the solution of (1.1.3) in Figure 1.2.

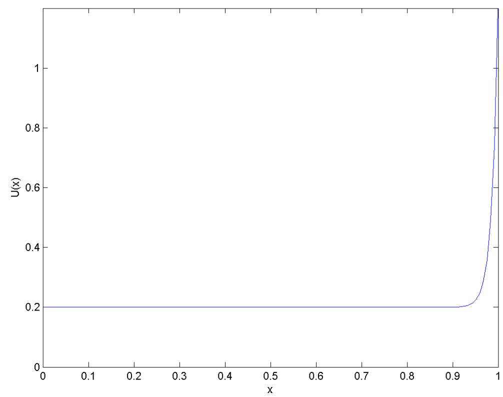

Figure 1.2: Solution of Example 1.1.2 showing the boundary layer near $x = 1$.

For a demonstration purpose we give a general two-point non-linear boundary value problem which has an interior layer.

Example 1.1.3.

$$-\varepsilon\frac{d^2u}{dx^2} + u\frac{du}{dx} + u = 0, \;\; u(-1) = u_{-1}, \;\; u(1) = u_1. \qquad (1.1.5)$$

When $\varepsilon = 0$, we have $v(x)v'(x) + v(x) = 0$, as the reduced problem with the solutions $v(x) = 0$ and $v(x) = -x + k$. Using the boundary conditions gives $v(x^-) = -x + u_{-1} - 1$ and $v(x^+) = -x + u_1 + 1$. Thus the layer will occur at the interior $(u_{-1} + u_1)/2$ of the domain. We display the interior layer in Figure 1.3.

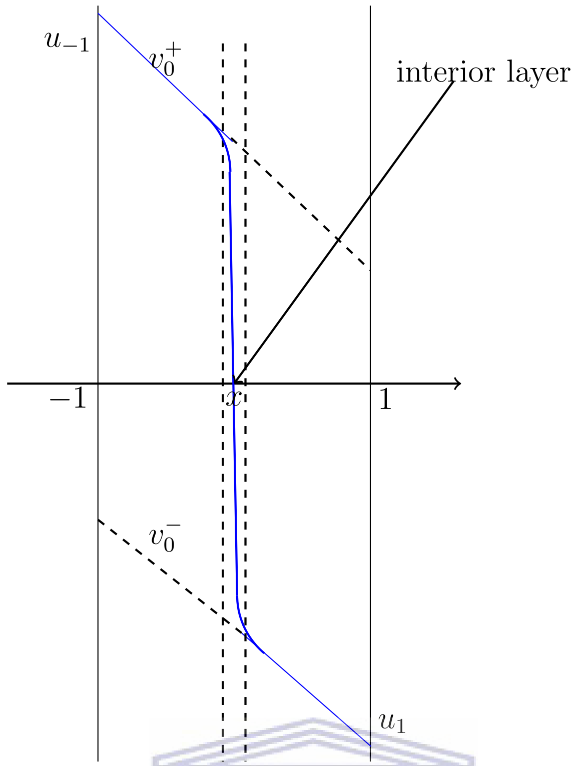

Figure 1.3: Solution of Example 1.1.3 with interior layers.

Since SPPs play a predominant role in applied sciences and engineering, finding their solutions is a necessity. However, in most cases, determining analytical solutions to such problems is difficult, if at all possible. Now the idea is to determine the best approximation to the analytical solution. The choice of strategies to determine such approximations is dictated by many factors. One important factor to be considered is the physical systems of concern [38] so that the physical properties of the solution is preserved after the problem has been solved. Two broad categories of these methods are the Asymptotic Methods and the Numerical Methods.

Asymptotic methods describe the qualitative behaviour such as the location and width of the layers in the solution. Examples of these asymptotic methods include, the Successive Complementary Expansions, Matched Asymptotic Expansion and the Method of Multiple Scales. The most used of these asymptotic methods are the Matched Asymptotic Expansion and the Method of Multiple Scales. For more insight on the use of asymptotic methods, interested readers are referred to [60, 47, 61, 62] and the references therein .

Numerical methods, on the other side provide the quantitative behaviour of the solution. These numerical methods include the Finite Difference, Finite Element, Finite volume and Spectral Methods. However, in SPPs, the classical numerical methods we mentioned earlier do not give satisfactory results as the singular perturbation parameter approaches zero, [16]. This is due to the fact that these classical numerical methods do not take into account the behaviour of the solutions in the layer regions. This leads to large errors when compared with the exact solutions, unless a large number of mesh points is used in the approximation process. However, this renders the numerical method computationally inefficient. Sometimes, the increase in mesh points also causes the resulting systems of algebraic equations to be ill conditioned. Therefore, there is a need for methods which are not prone to these computational difficulties and can serve as a better approximations of the exact solution. These methods are said to be $\varepsilon-$uniformly convergent and are defined in 1.1.1 according to the following definition .

Definition 1.1.1 ([34])**.** *Consider a family of mathematical problems parametrized by a singular perturbation parameter ε, where ε lies in the semi-open interval $0 < \varepsilon \leq 1$. Assume that each problem in the family has a unique solution denoted by u_ε is approximated by a sequence of numerical solutions $\{U_\varepsilon, \bar{\Omega}^n\}_{n=1}^{\infty}$, where U_ε is defined on the mesh $\bar{\Omega}$ and n is the discretization parameter. Then the numerical solutions U_ε are said to converge ε-uniformly to the exact solution u_ε, if there exist a positive integer n_0 and positive numbers C and p, where n_0, p and C are all independent of n and ε, such that, for all $n \geq n_0$,*

$$\sup_{0<\varepsilon\leq 1} ||U_\varepsilon - u_\varepsilon||_{\bar{\Omega}_\tau^n} \leq Cn^{-p}.$$

Here p is called the ε-uniform rate of convergence and C is called the ε-uniform error constant.

In this definition we used the maximum norm. In the rest of the thesis, whenever a norm is required we will use the discrete or the continuous maximum norm according to the situation at hand.

This dissertation is concerned with the design and implementation of $\varepsilon-$uniform convergent methods (according to Definition 1.1.1) in the context of finite difference schemes. Two classes of such schemes exist namely the fitted mesh finite difference methods (FMFDM) and the fitted operator finite difference methods (FOFDM).

Next we describe these two classes with more details.

1.2 Finite difference methods

In the context of SPPs, two categories of finite difference methods (FDMs) have been used by researchers. These FDMs, called fitted FDMs, are designed in such a way that they handle the numerical ”instabilities” created by the presence of the perturbation parameter.

Now, we show how the FOFDMs and the FMFDMs are designed for the reaction-diffusion and convection-diffusion problems.

1.2.1 Fitted operator finite difference methods

Fitted Operator Finite Difference Methods (FOFDMs) were introduced by Lubuma and Patidar ([30, 31]) by applying the modelling rules of Mickens which gave rise to the Non-standard Finite Difference Methods (NSFDMs) [32]. The FOFDMs consist of replacing the classical finite difference operator by one which captures the layer behaviour of the problem on a uniform mesh/grid.

In the two examples below, we see how these FOFDMs are designed in practice for time-dependent SPPs.

The case of a reaction-diffusion problem

We consider the problem in Example 1.1.1. According to [33], the concept of sub-equations is the major tool to derive the denominator function for a PDE. The denominator function in pivotal is the process of replacing the classical finite difference operator by a fitted operator. Thus we write (1.1.1) as

$$-\varepsilon\frac{d^2u}{dx^2}+bu=0,\;\;\frac{\partial u}{\partial t}-\varepsilon\frac{\partial^2 u}{\partial x^2}=0\;\;,\frac{du}{dt}+bu=0,$$

and design the scheme for each sub-equation. Let n and K denote positive integers. Consider the uniform sub-divisions of $\Omega = [0,1]$ and $[0,T]$ as follows:

$$x_0 = 0,\;\; x_i = x_0 + i\Delta x,\;\; i = 1(1)n-1,\;\; \Delta x = x_i - x_{i-1},\;\; x_n = 1.$$

$$t_0 = 0,\;\; t_k = t_0 + k\tau,\;\; k = 1(1)K-1,\;\; \tau = t_k - t_{k-1},\;\; t_K = T.$$

We denote the approximation of $u(x_i) \equiv u_i$ and $u(t_k)$ in the case of ODEs by U_i and U^k respectively. For the solution $u(x_i,t_k) = u_i^k$ of a PDE, we use the notation U_i^k. Using the theory of finite difference methods we obtain the schemes

$$\begin{aligned} -\varepsilon\frac{U_i - 2U_i + U_{i-1}}{\phi} + bU_i &= 0, \qquad (1.2.6)\\ \frac{U_i^{k+1} - U_i^k}{\tau} = \varepsilon\frac{U_i^k - 2U_i^k + U_{i-1}^k}{\phi}, &\qquad \frac{U^{k+1} - U^k}{\tau} + bU^k = 0. \end{aligned}$$

We calculate the denominator function ϕ in (1.2.6) as follows: the exact solution of the first equation is the linear combination of the terms $\exp\left(-\sqrt{b/\varepsilon}\Delta x\right)$ and $\exp\left(\sqrt{b/\varepsilon}\Delta x\right)$. Now we follow [32] to construct a second order difference equation as follows

$$\begin{vmatrix} U_i & U_{1;i} & U_{2;i} \\ U_{i+1} & U_{1;i+1} & U_{2;i+1} \\ U_{i+2} & U_{1;i+2} & U_{2:i+2} \end{vmatrix} = \begin{vmatrix} U_i & \exp\left(-\sqrt{\frac{b}{\varepsilon}}\Delta x_i\right) & \exp\left(\sqrt{\frac{b}{\varepsilon}}\Delta x_i\right) \\ U_{i+1} & \exp\left(-\sqrt{\frac{b}{\varepsilon}}\Delta x_{i+1}\right) & \exp\left(\sqrt{\frac{b}{\varepsilon}}\Delta x_{i+1}\right) \\ U_{i+2} & \exp\left(-\sqrt{\frac{b}{\varepsilon}}\Delta x_{i+2}\right) & \exp\left(\sqrt{\frac{b}{\varepsilon}}\Delta x_{i+2}\right) \end{vmatrix} = 0. \qquad (1.2.7)$$

Simplifying the determinant (1.2.7) and lowering the index i by one give the difference scheme

$$U_{i+1} - 2\cosh\left(\sqrt{\frac{b}{\varepsilon}}\Delta x_i\right)U_i + U_{i-1} = 0 \qquad (1.2.8)$$

But $\sinh(\Delta x/2) = \pm\sqrt{(\cosh(\Delta x) - 1)/2}$, thus from (1.2.6) and (1.2.8) we obtain the scheme

$$-\varepsilon\frac{U_{i+1} - 2U_i + U_{i-1}}{(4\varepsilon/b)\sinh^2(\sqrt{b/\varepsilon}\Delta x_i/2)} + bU_i = 0,$$

with the denominator function $\phi = (4\varepsilon/b)\sinh^2(\sqrt{b/\varepsilon}\Delta x_i/2)$. Now combining the sub-discrete difference schemes above we obtain

$$\frac{U_i^k - U_i^{k-1}}{\tau} - \varepsilon\frac{U_{i+1}^k - 2U_i^k + U_{i-1}^k}{\left(4\frac{\varepsilon}{b}\sinh\left(\sqrt{\frac{b}{\varepsilon}}\Delta x_i\right)\right)} + bU_i^k \;=\; f_i^k.$$

The case of a convection-diffusion problem

We consider the problem in 1.1.2 and follow the basic procedure in [33] to calculate the denominator function of the scheme.

Similarly, we write the sub-equations

$$\frac{\partial u}{\partial t} - \varepsilon \frac{\partial u}{\partial x^2} = 0, \quad \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad \frac{du}{dt} + bu = 0,$$
$$-\varepsilon \frac{d^2 u}{dx^2} + a\frac{du}{dx} = 0, \quad -\varepsilon \frac{d^2 u}{dx^2} + bu = 0, \quad a\frac{du}{dx} + bu = 0.$$

Following the same procedure as before and using the same notations we obtain the difference schemes for each sub-equation as

$$\frac{U_i^{k+1} - U_i^k}{\tau} = \varepsilon \frac{U_{i+1}^k - 2U_i^k + U_{i-1}^k}{\phi^2}, \quad \frac{U_i^{k+1} - U_i^k}{\tau} = -a\frac{U_i^k - U_{i-1}^k}{\Delta x}, \quad \frac{U^{k+1} - U^k}{\tau} - = bU^k$$
$$\varepsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{\phi} = a\frac{U_i - U_{i-1}}{\Delta x}, \quad \frac{U_{i+1} - 2U_i + U_{i-1}}{\phi^2} = bU_i, \quad a\frac{U_i - U_{i-1}}{\Delta x} = bU_i.$$

We calculate ϕ from the scheme

$$\varepsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{\phi} = a\frac{U_i - U_{i-1}}{\Delta x},$$

as follows: the sub-equation

$$-\varepsilon \frac{d^2 u}{dx^2} + a\frac{du}{dx} = 0,$$

can be rewritten as a system of two first order coupled differential equations as

$$\frac{du}{dx} = y, \quad \frac{dy}{dx} = \frac{a}{\varepsilon} y.$$

Now to obtain the discrete difference scheme for y we use the first order forward difference scheme

$$y_i = \frac{U_{i+1} - U_i}{\Delta x}.$$

The discrete form of dy/dx is given by

$$\frac{a}{\varepsilon} y_i = \frac{y_{i+1} - y_i}{\Delta x} \Rightarrow y_{i+1} = \frac{a\Delta x}{\varepsilon} y_i + y_i, \quad y_{i+1} = y_i \left(\frac{a\Delta x}{\varepsilon} + 1 \right).$$

Since $(a\Delta x/\varepsilon + 1)$ occurs in the finite difference scheme, we replace the denominator Δx by $\varepsilon a^{-1} \exp\left(a\Delta x/\varepsilon - 1\right)$. Combining these two equations and replacing the y gives

$$\frac{\varepsilon \Delta x}{a} \exp\left(\frac{a\Delta x}{\varepsilon} - 1 \right),$$

as the value of the denominator function ϕ_i. Combining the sub-schemes, we obtain the difference scheme

$$\frac{U_i^k - U_i^{k-1}}{\tau} - \varepsilon \frac{U_{i+1}^k - 2U_i^k U_{i-1}^k}{\phi_i} + a\left(\frac{U_i^k - U_{i-1}^k}{\Delta x}\right) + bU_i^k \quad = \quad f_i^k, \;\; \phi = \frac{\varepsilon \Delta x}{a} \exp\left(\frac{a\Delta x}{\varepsilon} - 1\right).$$

1.2.2 Fitted mesh finite difference methods

The family of Fitted Mesh Finite Difference Methods (FMFDMs) consist of discretizing the continuous problems via a classical finite difference scheme on a appropriately modified mesh. The mesh is modified in such a way that the method captures the difficulties inherent to the presence of the perturbation parameter. The meshes/grids employed are basically non-uniform and are of various types. There are piecewise uniform meshes (also known as meshes of Shishkin type) which are a union of two or more uniform meshes with different step-sizes, usually fine inside the layer(s) region(s) and coarse outside. There are also graded meshes which include Bakhvalov's and Vulanovic's meshes. These meshes are uniform outside the layer(s) region(s) and graded from very fine to coarse inside the layer(s) region(s). In either case, the location of the layers as well as their sizes must be determined before designing the meshes.

In this dissertation, reference to FMFDMs implies usage of piecewise uniform meshes. Now we show, in practice, how these meshes are designed.

Let Ω^n be the non-uniform discrete domain. Here, n is the number of mesh points, defined to satisfy $n > 2^r,\; r \geq 2$ and σ is the transition point which separates the layer region and the non-layer region. As indicated earlier problem (1.1.1)-(1.1.2) is characterized with two boundary layers. Thus we sub-divide the domain $[0, 1]$ into three sub-domains; $[0, \sigma]$, $[\sigma, 1-\sigma]$ and $[1-\sigma, 1]$, each with the step size $4\sigma/n$, $2(1-2\sigma)/n$ and $4\sigma/n$. That is $[0, \sigma]$, $[1-\sigma, 1]$, are for the respective layer regions near $x = 0$ and $x = 1$, and $[\sigma, 1-\sigma]$ for the non-layer region. The step size for the entire domain $[0, 1]$ is given by

$$\Delta x_i = x_i - x_{i-1} := \begin{cases} 4\sigma n^{-1}, & i = 1, 2, ..., n/4, \\ 2(1-2\sigma)n^{-1}, & i = n/4+1, ..., 3n/4, \\ 4\sigma n^{-1} & i = 3n/4+1, ..., n. \end{cases}$$

The transition parameter σ is given by

$$\sigma = \min\left\{\frac{1}{4}, \sigma_0\sqrt{\varepsilon}\ln n\right\}$$

and satisfies $0 < \sigma < 1/4$. Note that when $\sigma = 1/4$, then Shishkin mesh is a uniform mesh. Now we obtain the mesh points

$$\Omega^n = \{0 = x_0 < ... < x_n = 1\}.$$

Figure 1.4 below illustrates the Shishkin mesh for problem (1.1.1).

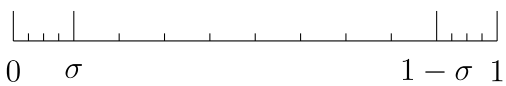

Figure 1.4: Shishkin mesh for $n = 16$ for the reaction-diffusion problem (1.1.1).

Problem (1.1.3)-(1.1.4) has a single boundary layer thus we divide the interval $[0, 1]$ into $[0, 1-\sigma]$ and $[1-\sigma, 1]$ each with the step size $n/2$ and $n/2+1$ mesh points. For this problem σ lies in the interval $0 < \sigma < 1/2$ and it is given by

$$\sigma = \min\left\{\frac{1}{2}, \sigma_0\varepsilon\ln n\right\}.$$

Also, when $\sigma = 1/2$, then the mesh is a uniform mesh. We obtain the mesh width

$$\Delta x_i = x_i - x_{i-1} := \begin{cases} 2(1-\sigma)n^{-1}, & i = 1, 2, ..., n/2, \\ 2\sigma n^{-1}, & i = n/2+1, ..., n. \end{cases}$$

Figure 1.5 illustrates Shishkin mesh for the spatial domain in problem (1.1.3)-(1.1.4).

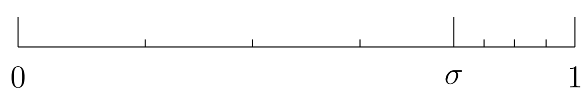

Figure 1.5: Shihskin mesh for a layer near $x = 1$.

Similarly when the layer is at the left of the domain Ω we use the mesh width

$$\Delta x_i = x_i - x_{i-1} := \begin{cases} 2\sigma n^{-1}, & i = 1, 2, ..., n/2 \\ 2(1-\sigma)n^{-1}, & i = n/2+1, ..., n, \end{cases}$$

with σ defined as before.

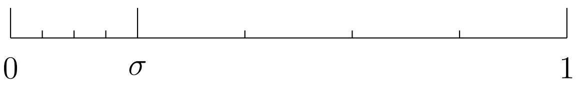

Figure 1.6: Shihskin mesh for a layer near $x = 0$.

In this thesis we employ the method of lines to approximate singularly perturbed problems. Next, we discuss the method of lines.

1.3 Method of Lines

A two-stage approach where the independent variables in the initial-boundary value problem are decoupled, solved and analysed separately is what is termed as Method of Lines (MOL). More specifically, the spatial domain is discretized and at the same time the spatial differential operator is replaced with a discrete difference operator. Usually this depends on the order of the spatial differential operator and the choice of numerical method intended for the spatial approximation. At this stage the time variable is held continuous on it domain. This results in systems of initial value problems with incorporated boundary conditions known as Differential Algebraic Equations (DAE). Solutions of these DAEs can be obtained with standard integrators as described in [1] or can still be discretized to achieve the desired result. Actually one major advantage of using this approach is the many options one has on the availability of vast sophisticated software's for integrating the DAEs. This approach is sometimes called the longitudinal method of Lines [17].

Another school of thought suggests that the Method of Lines can be viewed as a continuous spatial domain with a discrete time domain. Hence the time differential operator is replaced with a discrete difference operator whilst the spatial differential operator is maintained, resulting in boundary value problems. For instance elliptic problems will be the outcome of parabolic problems with more than one spatial variables after the time discretization. From here the resulting DAEs which is a combination of boundary value problems and initial conditions can be solved with boundary value methods like the fitted operator finite difference methods, the moving mesh methods or even the finite element methods. This is known as the transversal method of lines [17] or the Rothe's method [54]. We refer to these two approaches here and there after as the method of lines and the Rothe's method.

These two approaches are sometimes called the semi-discrete methods either in space for the former or in time for the latter. Both methods have the advantage of being computationally efficient over the methods which discretize all the independent variables at the same time. This is due to the fact that they reduce computational effort, time as well as the cost of computations. All because it allows the computation of one independent variable at a time. Also, with this semi-discrete approach a higher order PDE with more than one spatial variables can be solved computationally without any difficulties. Hence they are sometimes referred to as the standard tools or approaches for solving complex and practical electromagnetic problems [56].

In the second part of the MOL, the ODEs are then integrated in time. In the selection of an ODE solver properties such as accuracy and computational complexities are considered so that the physical properties of the PDE are preserved. Usually, the spectrum of the discrete spatial operator serves as a guide in the selection.

Remark 1.3.1. *Note that the effectiveness of this method requires that the eigenvalues of the spatial discrete operator scaled by the time step should lie in the stability region of the ODE solver.*

There are several numerical techniques for solving these ODEs. These include the Euler method, the Midpoint Rule, Crank Nicholson's method, Runge Kutta methods, etc. In this thesis we employ the implicit Euler method to integrate the IVPs which result from the spatial discretization. This method is known to be a strong stability preserving numerical method. A numerical method which enjoys the following properties is a strong stability preserving method [18].

a. Monotonicity

If u^m and u^{m-1} are the solutions at the times t^m and t^{m-1} times respectively, then it holds that either

$$||u^m|| \leq ||u^{m-1}|| \text{ or } ||u^{m-1}|| \leq ||u^m||,$$

b. Contractivity

Given two approximate solutions u^{m-1} and $\tilde{u}^{m-1}$ at the time t^{m-1}, and u^m and $\tilde{u}^m$ at the time t^m. Where these two solutions represent the solution of the original problem and a perturbed problem we have

$$||u^m - \tilde{u}^m|| \leq ||u^{m-1} - \tilde{u}^{m-1}||,$$

or vice versa.

c. Positivity

$$u^m \geq 0, \ \forall \ t^m$$

If the numerical method admits the above properties then it also satisfies the absolute monotonicity condition

$$||\mathbf{I} + \tau\mathcal{L}|| \leq 1. \tag{1.3.9}$$

Below we demonstrate the method of lines in practice.

1.3.1 Case 1: Example 1.1.1

Spatial discretization

We suppose $\varepsilon = 1$ and partition the domain Ω as follows:

$$x_0 = 0, \ \ x_i = x_0 + i\Delta x, \ \ \Delta x = x_i - x_{i-1}, \ \ i = 1(1)n, \ \ x_n = 1, \tag{1.3.10}$$

where Δx is the step size and n is the number of sub-intervals. We denote the approximation of $u(x_i, t) \equiv u_i(t)$ via the spatial discretization by $U(x_i, t) \equiv U_i(t)$. The discretization in space yields the semi-discrete problem

$$\mathcal{L}^n U_i(t) \equiv \frac{d}{dt} U_i(t) - \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{\Delta x^2} + b_i(t) U_i(t) = f_i(t), \tag{1.3.11}$$

with the semi-discrete boundary and initial conditions $U_0(t) = \eta_0,\ U_n(t) = \eta_1$ and $U_i(0) = \varphi_i,\ i = 1, 2, ..., n-1$. Equation (1.3.11) can be written in the matrix notation as

$$U'(t) + A(t)U(t) = F(t), \tag{1.3.12}$$

with the initial condition $U_i(0) = \varphi_i,\ i = 1, ..., n-1$, where $F(t) \in \mathbb{R}^{n-1}$, is the semi-discrete form of $f_i(t)$ and the boundary conditions, $U(t) \in \mathbb{R}^{(n-1)}$ and $A(t) \in \mathbb{R}^{(n-1)\times(n-1)}$. The entries of these variables are given as

$$\begin{aligned}
A_{ii}(t) &= 2\Delta x^{-2} + b_i(t), \quad & i = 1, 2, ..., n-1,\\
A_{i,i+1}(t) &= -\Delta x^{-2}, & i = 1, 2, ..., n-2,\\
A_{i,i-1}(t) &= -\Delta x^{-2}, & i = 2, 3, ..., n-1,\\
F_1(t) &= f_1(t) + \eta_0 \Delta x^{-2}, &\\
F_i &= f_i(t), & i = 2, 3, ..., n-2,\\
F_{n-1}(t) &= f_{n-1}(t) + \eta_1 \Delta x^{-2}. &
\end{aligned}$$

It is to be noted that the coefficient matrix $A(t)$, is usually sparse and has a banded structure which depends on the finite difference formula used for the spatial discretization. Thus A is a square triadiagonal matrix.

The second order central difference approximation is known to be stable and consistent of order two.

IVP-integration

We integrate equation (1.3.12) on a uniform mesh. Here we take advantage of the shelf solvers. We consider an example in [13] to show this numerically. We take $b = 1,\ f = 0,\ u(0,t) = u(1,t) = 0,\ u(x,0) = \sin(\pi x)$ and the exact solution is $u(x,t) = \exp(-(\pi^2+1)t)\sin(\pi x)$. We compute the maximum pointwise error and the numerical rate of convergence. The exact solution of this problem is given, thus we compute the error with the formula

$$E_n = \max_{(x_i,t_k)\in Q^{n,K}} |U(x_i,t_k) - u(x_i,t_k)|, \tag{1.3.13}$$

where $U(x_i, t_k)$ is the numerical solution and $u(x_i, t_k)$ is the exact solution. To compute the numerical rate of convergence we use the formula

$$r = \log_2 \left(E_n / E_{2n} \right). \tag{1.3.14}$$

Table 1.1: Maximum pointwise error and rate of convergence for Problem (1.1.1)-(1.1.2) via MOL as presented above.

n	E_n	r
16	$1.10E-03$	2.04
32	$2.67E-04$	2.05
64	$6.65E-05$	2.04
128	$1.62E-05$	2.14
256	$3.67E-06$	2.75
512	$5.45E-07$	

1.3.2 Case 2: Example 1.1.2

Following the same procedure as previously and using the partition (1.3.10) we have the following discretization

$$\begin{aligned} \mathcal{L}^n U_i(t) \equiv & \frac{d}{dt} U_i(t) - \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{\Delta x^2} + a_i(t) \left(\frac{U_i(t) - U_{i-1}(t)}{\Delta x} \right) \\ & + b_i(t) U_i(t) = f_i(t), \end{aligned} \tag{1.3.15}$$

with the semi-discrete boundary and initial conditions $U_0(t) = \eta_0 \;\; U_n(t) = \eta_1$, and $U_i(0) = \varphi_i, \;\; i = 1, 2, ..., n-1$. We write equation (1.3.15) in the matrix notation

$$U'(t) + A(t)U(t) = F(t), \tag{1.3.16}$$

with the initial condition $U_i(0) = \varphi_i, \;\; i = 1, 2, ..., n-1$. Where $U(t), \;\; F(t) \in \mathbb{R}^{n-1}$, and $A(t) \in \mathbb{R}^{(n-1)\times(n-1)}$. Their entries are given as

$$A_{ii}(t) = 2\Delta x^{-2} + a_i(t)\Delta x^{-1} + b_i(t), \qquad i = 1, 2, ..., n-1,$$

$$
\begin{aligned}
A_{i,i+1}(t) &= -\Delta x^{-2}, & i = 1, 2, ..., n-2,\\
A_{i,i-1}(t) &= -\Delta x^{-2} - a_i(t)\Delta x^{-1}, & i = 2, 3, ..., n-1,\\
F_1(t) &= f_1(t) + (\Delta x^{-2} + a_1(t)\Delta x^{-1})\eta_0, &\\
F_i &= f_i(t), & i = 2, 3, ..., n-2,\\
F_{n-1}(t) &= f_{n-1}(t) + \Delta x^{-2}\eta_0. &
\end{aligned}
$$

Similar to the previous example we integrate the IVPs (1.3.16) with the built a built-in MATLAB integrators. In our computations, we take $\Omega = [-2, 2]$, $a = 1$, $b = 0$, $T = 4$, and the initial condition $u(x, 0) = \exp(-x^2)$. The boundary conditions are such that the exact solution is given by $(\sqrt{1+4t})^{-1}\exp(-(x-t)^2/(1+4t))$.

Table 1.2: Maximum pointwise error and rate of convergence for Problem (1.1.3)-(1.1.4) via MOL as presented above.

n	E_n	r
32	$6.51E-02$	0.88
64	$3.53E-02$	0.96
128	$1.82E-02$	0.98
256	$9.26E-03$	0.99
512	$4.67E-03$	0.99
1024	$2.35E-03$	

1.4 Literature review on numerical methods for time-dependent singularly perturbed problems

Quite often, it is difficult, if at all possible, to determine the exact solution of singularly perturbed partial differential equations that model real life situations. From the theory of differential equations, it is always possible to establish existence and uniqueness of such solution even if they cannot be calculated analytically. Numerical methods are therefore necessary to provide approximations to the solution. However, the challenges that face numerical analysts is to be able to design numerical methods which produce better approximations. It is in that context that standard numerical methods are inadequate for SPPs. In an attempt to resolve this issue, several works were published since the Third International Congress of Mathematicians held in Heidelberg in 1904 [51]. We now give a brief account on some of these works accomplished in the last two decades.

Linß [28] studied a time-dependent reaction-diffusion problem. He employed the classical finite element method on a layer adapted mesh to approximate the spatial derivative along with the backward Euler for the time discretization. The method was shown to be of first order accuracy in time and second order in space.

Miller et al. [35] used the backward Euler method along with the classical finite difference method on a piecewise uniform mesh to approximate the time and the space derivatives. Their analysis which made use of solution decomposition and special barrier functions gave a first order accuracy and almost second order accuracy in time and space. Using the Green functions, Linß and Madden [29] provided a more general analysis the methods designed in [35] and showed that the method is first order accurate in time and almost second order accurate in space.

Clavero and Gracia [9] combined the backward Euler and the classical finite difference method on a layer adapted mesh for their approximation. They resorted to three different meshes namely the Shishkin, the Bahkvalov and the Vulanovic for the spatial discretiz- tion. They analysed their schemes for convergence and obtained an almost second order accuracy for the Shishkin and the Vulanovic meshes, with that of the Bakhavalov mesh yielding a second order accuracy.

Natesan and Deb in [46] proposed a numerical method which is ε-uniform of order $\mathcal{O}(n_x^{-2}\ln^2 n_x + n_t^{-1})$. Here n_x and n_t are the number of sub-intervals in the space and time variables respectively. They employed a hybrid scheme on a piecewise uniform mesh to discretize the spatial variable along with the backward Euler for the time derivative. Note that their hybrid scheme was a combination of the cubic spline which approximated the boundary layer part and the central difference approximation which also dealt with the non layer part.

In [43], Munyakazi and Patidar proposed a discretisation of a reaction diffusion poblem in which they used the fitted operator finite difference method to approximate the spatial derivative and the backward Euler for the time derivative. This scheme was shown to be second and first order accurate in space and time, respectively.

The variational-perturbation theory was used by Zhou and Wu in [63] to integrate a time-dependent reaction-diffusion problem exhibiting both boundary layer and outer regions. The authors showed that their method is accurate.

Kopteva and Savscu [23] considered a time-dependent semi-linear reaction-diffusion problem. They employed both Bakhavalov and Shishkin types of meshes for the spatial discretization together with the backward Euler method for the time discretization. Using the discrete upper and lower solutions they obtained the error bounds $C(\tau + n^{-2}\ln^2)$ and $C(\tau + n^{-2})$ for the Shishkin and the Bakhavalov mesh respectively. Here and thereafter τ is the width of the mesh spacing in time and $\varepsilon,\ n$ are as defined earlier.

In the case of systems of time-dependent reaction-diffusion problems, Gracia and Lisbona [19] used the implicit Euler method and the classical finite difference method on a piecewise uniform mesh for the time and space discretizations. They analysed their scheme for convergence and obtained a second order accuracy in space and a first order in time.

Ramos [52] studied a time-dependent convection-diffusion problems. He used an exponentially fitted finite difference method to discretize the spatial variable and the backward Euler for the time variable. His analysis gave a first order accuracy in both space and time variables.

Lenferink [25] considered a time-dependent convection-diffusion problem in the framework of method of lines. He employed the classical finite element method on a Bakhavalov-Shishkin type of meshes to approximate the spatial derivatives and the implicit midpoint rule for the time integration. Analysis of both discretization gave a second order accuracy in both space and time variables.

Ng-Stynes et al. [45] used the semi-discrete Petrov-Galerkin finite element method to integrate a time-dependent convection-diffusion problem with variable coefficients. Their analysis gave a first order accuracy in both variables. Kadalbajoo et al. [22] studied the same problem via the Rothe's method. They used the standard implicit finite difference scheme for the temporal discretization and the B-spline collocation method on a piecewise uniform mesh for the spatial discretization. They showed that their method is first order accurate in time and second in space.

Similarly, Kadalbajoo and Awasthi [21] developed a parameter uniform numerical method to approximate a time-dependent convection-diffusion problem. They combined the Crank Nicholson finite difference method and the second order upwind scheme on a piecewise uniform mesh to approximate the time and space variables respectively. Their analysis gave a second order accuracy in time and an almost first order in space.

Clavero and Gracia [5] also studied a one-dimensional time-dependent convection-diffusion problem. They used the implicit Euler method for the time discretization and the simple upwind scheme on a special non-uniform mesh for the spatial discretization. They proved that their method is of first order accuracy in time and almost first order in space.

Still on the same problem, Natesan and Gowrisankar [44] used the backward Euler and the classical upwind finite difference method on a layer adapted mesh for the time and space variables respectively. Their analysis gave a first order accuracy in time and almost first order in space.

In [2], Cheng and Liu used a positive monitor function to develop an adaptive grid for the spatial discretization and the backward Euler for the time discretization. The analysis of each discretization gave a first order uniformly convergent rate.

Rao and Srivastava [53] treated a time-dependent weakly coupled linear system of singularly perturbed convection-diffusion equations. These authors combined the backward Euler method on a uniform mesh and the HODIE (high order differences with identity expansion) scheme together with the classical finite difference scheme on Shishkin mesh to discretize the time and space variables respectively. Their analysis gave a first order accuracy in time and an almost second order in space. Munyakazi [39] studied a two parameter convection-diffusion problem via the Rothe's method. He used the backward Euler method for the time discretization and the FOFDM for the space discreization. His analysis gave a first order accuracy in both space and the time variables. Still on the two parameters affecting the first and the second spatial derivatives, Miller et al. [36] constructed a monotone finite difference method on a piecewise uniform mesh of first order in both variables except for a logarithmic factor in the spatial variable.

For a two-dimensional reaction-diffusion problems, Clavero et al. [4] used the alternating direction method and the classical finite difference method on a non-uniform mesh to discretize the time and the space variables respectively. Their analysis gave a first order accuracy in both variables.

Clavero et al. [3] discretized the spatial variables of a convection-diffusion problem with the classical upwind finite difference method on non-uniform mesh and the time variable with the fractional step method. Their analysis gave a first order accuracy in time and an almost first order accuracy in space. In [8], Clavero et al. used the Peaceman and Rachford methods to descretize the time variable and HODIE (high order differences with identity expansion) finite difference method to discretize the space variables. The authors proved their method to be second order accurate in time and an almost second order accurate in space. In [12], Clavero and Jorge treated both two-dimensional convection and reaction-diffusion time-dependent problems. The spatial discretization of both problems were done with the FMFDM particularly on a Shishkin mesh and the implicit Euler integrating method on a uniform mesh was used for the time discretization. Their analysis resulted in an almost first order accuracy in space for the convection-diffusion problem and an almost second order accuracy in space for the reaction-diffusion problem. Both with respect to the perturbation parameter and the implicit Euler also yielding a first order accuracy.

From the above, we observe that a large amount of work has been done on time-dependent singularly perturbed problems as far as designing and analysing numerical methods for their integration is concerned. In most of these works, FMFDMs have been adopted. Moreover, the method of lines has received very little attention from the research community. In this thesis we consider time-dependent singularly perturbed problems. We will solve problems in the framework of the method of lines. On one side we will review the existing FMFDMs and on the other, we will design, analyse and implement FOFDMs.

1.5 Objectives and organisation

The method of lines involves a step by step discretization. First, the spatial variable is discretized followed by the discretization of the time variable. Discretization of the spatial variable results in a system of semi-discrete problems. These are in turn discretized in time via the backward Euler method.

We will review some existing methods where the spatial discretization is done via FMFDMs. In this case we will refer to the method as the Fitted Mesh Finite Difference Method of lines (FMFDMLs). We will also design methods where the discretization in space is performed via FOFDMs. The resulting scheme will be termed the Fitted Operator Finite Difference Method of Lines (FOFDMLs).

The summary of each chapter is as follows. In Chapter 2, we consider a time-dependent reaction-diffusion problem. The chapter begins with some qualitative properties of the continuous problem whose replication in the semi-discrete form are used to analyse the numerical methods in later sections. Next, we design fitted numerical methods (FMFDM and FOFDM) to integrate the PDE in space. Subsequently, we provide some properties of the semi-discrete problem which are then used to give a detailed analysis of each spatial discretization. We integrate the IVPs with the backward Euler method and analyse it for convergence. To support the analysis we perform numerical experiments with a text example.

Following the procedures in Chapter 2, we study a convection dominated one-dimensional time-dependent problem in Chapter 3. Chapter 4 treats a two-dimensional time-dependent reaction-diffusion whilst its corresponding convection-diffusion problem is studied in Chapter 5.

In the final Chapter, we give a brief discussion of the thesis with some concluding remarks of the whole picture as well as future direction of this research.

Chapter 2

Methods of Lines for One-Dimensional Reaction-Diffusion Problems

In the previous chapter we gave an overview of singularly perturbed problems and the method of lines. Also, we introduced some ε-uniform numerical methods to solve such problems. Our objective in this chapter is to use the FMFDM of lines and the FOFDM of lines for time-dependent singularly perturbed reaction-diffusion problem. First, we discretize the space variable with a FMFDM and analyse it for convergence. Then we do the same with a FOFDM. The resulting systems of initial value problems for each method is then solved with the backward Euler integration technique. To illustrate the method in practice, we integrate a test example.

2.1 Continuous problem

We consider the one-dimensional time-dependent reaction-diffusion problem

$$\mathcal{L}u(x,t) \equiv u_t - \varepsilon u_{xx} + b(x,t)u = f(x,t), \quad (x,t) \in Q = \Omega \times (0,T],\ \Omega = (0,1), \quad (2.1.1)$$

with the boundary and initial conditions

$$u(0,t) = \eta_0, \quad u(1,t) = \eta_1, \ \text{ and } \ u(x,0) = \varphi(x),\ \ x \in \bar{\Omega},\ \ t \in [0,T], \qquad (2.1.2)$$

where η_0 and η_1 are given constants and $0 < \varepsilon << 1$. The functions $b(x,t)$ and $f(x,t)$ are assumed to be sufficiently smooth such that $b(x,t) \geq \beta > 0,\ \forall\ (x,t) \in Q$ and $b, f \in \mathcal{C}^{4,2}(\bar{Q})$. Also we impose the compatibility conditions

$$f(0,0) = f(1,0) = 0,\ f_{xx}(0,0) + b(0,0)f(0,0) = f_t(0,0),\ f_{xx}(1,0) + b(1,0)f(1,0) = f_t(1,0),$$

for the solution of problem (2.1.1)-(2.1.2) to be compatible at the corners of Q. Setting $\varepsilon = 0$, we obtain the reduced problem

$$\mathcal{L}u^0 \equiv u^0_t + b(x,t)u^0 = f^0, \quad (x,t) \in Q, \tag{2.1.3}$$

$$u^0(x,0) = \varphi(x),\ u^0(0,t) = \eta_0,\ u^0(1,t) = \eta_1. \tag{2.1.4}$$

The reduced problem (2.1.3)-(2.1.4) is an Initial Value Problem (IVP) which has two boundary conditions and an initial condition. Integration of this IVP will not make use of the two boundary conditions. As a result, there will be two boundary layers at the ends of the spatial domain each of width $\mathcal{O}(\sqrt{\varepsilon}|\ln \varepsilon|)$, see [11]. Note that these boundary layers are of parabolic type. The differential operator

$$\mathcal{L} = \frac{\partial}{\partial t} - \varepsilon \frac{\partial^2}{\partial x^2} + bI$$

admits the continuous maximum principle which ensures the stability of the solution. Next we provide qualitative properties of the solution to problem (2.1.1)-(2.1.2) with its derivatives.

Qualitative properties of the continuous problem

We follow [34] to present some properties of the continuous problem. These properties ensure the existence and uniqueness of the solution to problem (2.1.1)-(2.1.2).

Lemma 2.1.1. *(Continuous maximum principle). Let ξ be a sufficiently smooth function defined on Q which satisfies $\xi(x,t) \geq 0,\ \forall\ (x,t) \in \partial Q$. Then $\mathcal{L}\xi(x,t) > 0,\ \forall\ (x,t) \in Q$ implies that $\xi(x,t) \geq 0,\ \forall\ (x,t) \in \bar{Q}$.*

Proof. Let (x^*, t^*) be such that

$$\xi(x^*, t^*) = \min_{(x,t) \in \bar{Q}} \xi(x,t)$$

and suppose $\xi(x^*, t^*) < 0$. It is clear that $\xi(x^*, t^*) \notin \partial Q$. We have

$$\mathcal{L}\xi(x^*, t^*) = \xi_t(x^*, t^*) - \varepsilon\xi_{xx}(x^*, t^*) + b(x^*, t^*)\xi(x^*, t^*).$$

Since $\xi_{xx}(x^*, t^*) \geq 0$ and $\xi_t(x^*, t^*) = 0$, we obtain $\mathcal{L}(x^*, t^*) < 0$, which contradicts with the initial assumption that $\mathcal{L}(x, t) > 0, \ \forall \ (x, t) \in Q$. Therefore, $\xi(x, t) \geq 0, \ \forall \ (x, t) \in \bar{Q}$. □

Lemma 2.1.2. *(Stability estimate). Let $u(x,t)$ be the solution of the continuous problem (2.1.1)-(2.1.2). Then we have the bound*

$$||u|| \leq \beta^{-1}||f|| + \max(\varphi(x), \max(\eta_0, \eta_1)).$$

Proof. We define two comparison functions $\Psi^{\pm}$ as

$$\Psi^{\pm}(x, t) = \beta^{-1}||f|| + \max(\varphi(x), \max(\eta_0, \eta_1)) \pm u(x, t).$$

At the initial stage we have

$$\begin{aligned} \Psi^{\pm}(x, 0) &= \beta^{-1}||f|| + \max(\varphi(x), \max(\eta_0, \eta_1)) \pm u(x, 0) \\ &= \beta^{-1}||f|| + \max(\varphi(x), \max(\eta_0, \eta_1)) \pm \varphi(x) \\ &\geq 0, \end{aligned}$$

at the boundaries we obtain

$$\begin{aligned} \Psi^{\pm}(0, t) &= \beta^{-1}||f|| + \max(\varphi(0), \max(\eta_0, \eta_1)) \pm u(0, t) \\ &= \beta^{-1}||f|| + \max(\varphi(0), \max(\eta_0, \eta_1)) \pm \eta_0 \\ &\geq 0, \\ \Psi^{\pm}(1, t) &= \beta^{-1}||f|| + \max(\varphi(1), \max(\eta_0, \eta_1)) \pm u(1, t) \\ &= \beta^{-1}||f|| + \max(\varphi(1), \max(\eta_0, \eta_1)) \pm \eta_1 \\ &\geq 0, \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}\Psi^{\pm}(x, t) &= \Psi^{\pm}_t(x, t) - \varepsilon\Psi^{\pm}_{xx}(x, t) + b(x, t)\Psi^{\pm}(x, t) \\ &= b(x, t)(\beta^{-1}||f|| + \max(\varphi(x), \max(\eta_0, \eta_1))) \pm \mathcal{L}u(x, t) \\ &= b(x, t)(\beta^{-1}||f|| + \max(\varphi(x), \max(\eta_0, \eta_1))) \pm f(x, t) \\ &\geq 0, \quad \text{since} \quad b(x, t) > \beta. \end{aligned}$$

Therefore $\Psi^{\pm}(x,t) \geq 0, \;\; \forall\, (x,t) \in \bar{Q}$. This ends the proof. $\square$

To be able to carry out a fully fledge analysis of the numerical methods we will see in later sections, the bounds on the solution and its derivatives are needed. To obtain these bounds we follow [35].

These authors proved that under the smoothness and the compatibility conditions imposed on problem (2.1.1)-(2.1.2), its exact solution and its derivatives satisfy the bound

$$\left\| \frac{\partial^{i+k} u_\varepsilon}{\partial x^i \partial t^k} \right\| \leq C\varepsilon^{-\frac{i}{2}},$$

where i, k are integers which lie in the interval $0 \leq i + 2k \leq 4$ and C is a constant independent of ε. However, this bound cannot be used to obtain the ε-uniform bound of the numerical method we will see in later sections. Thus to obtain the ε-uniform bound we write the exact solution $u(x,t)$ as the sum

$$u(x,t) = v(x,t) + w(x,t),$$

where $v(x,t)$ and $w(x,t)$ are the regular and the layer components of $u(x,t)$, respectively. The regular component is the solution to the problem

$$\mathcal{L}v = f, \;\; (x,t) \in Q, \quad v = 0, \quad \text{on} \quad x \in \Omega, \;\; t = 0,$$

$$v = v^0, \quad \text{on } x \in \Omega, \;\; t \in \; (0,T].$$

Furthermore, we split v into the components

$$v(x,t) = v^0(x,t) + \varepsilon v^1(x,t),$$

where $v^0(x,t)$ is the solution of the reduced problem and $v^1(x,t)$, satisfies the equation

$$\mathcal{L}v^1(x,t) = \frac{\partial^2 v^0}{\partial x^2}, \quad (x,t) \in Q, \quad v^1 = 0, \;\; x \in (0,1), \;\; t \in (0,T].$$

Also, we have the layer part of the solution to satisfy the homogeneous problem

$$\mathcal{L}w = 0, \;\; (x,t) \in Q, \;\; w = 0, \;\; \text{on } x \in \Omega, \;\; t = 0.$$

$$w = -v^0, \;\; \text{on } x \in \Omega, \;\; t = (0,T].$$

Similar to the regular component, we divide $w(x,t)$ into

$$w_l(x,t) + w_r(x,t),$$

with the definitions

$$\mathcal{L}w_l = 0, \quad (x,t) \in Q, \; w_l = -v^0, \; x = 0, \; t \in (0,T], \; w_l = 0, \; x = 1, \; t \in (0,T],$$
$$\mathcal{L}w_r = 0, \quad (x,t) \in Q, \; w_r = -v^0, \; x = 1, \; t \in (0,T], \; w_r = 0, \; x = 0, \; t \in (0,T],$$

respectively. Clearly, w_l and w_r are the respective boundary layers at $x = 0$ and $x = 1$. To obtain the bounds of each component, recall that v^0 is the solution of the reduced problem, thus it is independent of ε and hence, satisfies the bound

$$\left\| \frac{\partial^{i+k} v^0}{\partial x^i \partial t^k} \right\|_{\bar{Q}} \leq C.$$

Also, v^1 is a solution to a problem of the same manner as the original problem and so has the bound

$$\left\| \frac{\partial^{i+k} v^1}{\partial x^i \partial t^k} \right\|_{\bar{Q}} \leq C\varepsilon^{\frac{i}{2}}.$$

In addition, we have

$$\begin{aligned} \left\| \frac{\partial^{i+k} v}{\partial x^i \partial t^k} \right\| &= \left\| \frac{\partial^{i+k} v^0}{\partial x^i \partial t^k} \right\| + \varepsilon \left\| \frac{\partial^{i+k} v^1}{\partial x^i \partial t^k} \right\|_{\bar{Q}}, \\ &= C + C\varepsilon\varepsilon^{-\frac{i}{2}}, \\ &\leq C(1 + \varepsilon^{(2-i)/2}), \end{aligned}$$

as the bound of the regular component of the solution. The bound of the left layer function is given as

$$\left\| \frac{\partial^{i+k} w_l(x,t)}{\partial x^i \partial t^k} \right\| \leq C\varepsilon^{-\frac{i}{2}} \exp\left(\frac{-x}{\sqrt{\varepsilon}} \right).$$

To prove this bound, we define two comparison functions $\Psi^{\pm}$ as

$$\Psi^{\pm}(x,t) = C \exp\left(\frac{-x}{\sqrt{\varepsilon}} \right) \exp(\alpha t) \pm w_l(x,t).$$

At the boundaries and the initial stages we have

$$\Psi^{\pm}(0,t) = C \exp\left(\frac{-0}{\sqrt{\varepsilon}} \right) \exp(\alpha t) \pm w_l(0,t)$$

$$
\begin{aligned}
&= C\exp(\alpha t) \pm v^0 \\
&\geq 0, \\
\Psi^{\pm}(1,t) &= C\exp\left(\frac{-1}{\sqrt{\varepsilon}}\right)\exp(\alpha t) \pm w_l(1,t) \\
&= C\exp\left(\frac{-1}{\sqrt{\varepsilon}}\right)\exp(\alpha t) \pm 0 \\
&\geq 0, \\
\Psi^{\pm}(x,0) &= C\exp\left(\frac{-x}{\sqrt{\varepsilon}}\right)\exp(\alpha 0) \pm w_l(x,0) \\
&= C\exp\left(\frac{-x}{\sqrt{\varepsilon}}\right) \pm 0 \\
&\geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{L}\Psi^{\pm}(x,t) &= \Psi_t^{\pm} - \varepsilon\Psi_{xx}^{\pm} + b(x,t)\Psi^{\pm} \\
&= [\alpha - 1 + b(x,t)]\exp\left(-x/\sqrt{\varepsilon}\right)\exp(\alpha t) \pm \mathcal{L}w_l \\
&= [\alpha - 1 + b(x,t)]C\exp\left(\frac{-x}{\sqrt{\varepsilon}}\right)\exp(\alpha t) \\
&\geq [\alpha - 1 + \beta]C\exp\left(\frac{-x}{\sqrt{\varepsilon}}\right)\exp(\alpha t) \\
&\geq 0.
\end{aligned}
$$

From the maximum principle the left layer function satisfies the bound

$$
\begin{aligned}
||w_l(x,t)|| &\leq C\exp\left(-\frac{x}{\sqrt{\varepsilon}}\right)\exp(\alpha t) \\
&\leq C\exp\left(\frac{-x}{\sqrt{\varepsilon}}\right)\exp(\alpha T) \\
&\leq C\exp\left(\frac{-x}{\sqrt{\varepsilon}}\right).
\end{aligned}
$$

To obtain the bounds on the derivatives of w_l we set $\tilde{x} = x/\sqrt{\varepsilon}$ so that the left layer function becomes the solution of

$$
\tilde{w}_{lt} - \varepsilon\tilde{w}_{lxx} + \tilde{b}w_l = 0, \;\; \tilde{Q} = \left(0, \frac{1}{\sqrt{\varepsilon}}\right) \times (0,T].
$$

In [24], the authors showed that with respect to the position of the stretched function $\tilde{x}$ two distinct cases are considered, one for the case when $\tilde{x}$ is in $(0,2] \times (0,T]$ and the other is when $\tilde{x}$ lies in the neighbourhood $(2, 1/\sqrt{\varepsilon}) \times (0, T]$. When $\tilde{x}$ is in $(2, 1/\sqrt{\varepsilon}) \times (0, T]$ then we have

$$||\tilde{w}_l(\tilde{x}, t)|| \le C||\tilde{w}_l||, \ \tilde{x} \in (0, 2] \times (0, T],$$

which results in

$$\left|\left|\frac{\partial^{i+k} w_l}{\partial x^i \partial t^k}\right|\right| \le C \exp\left(\frac{-x}{\sqrt{\varepsilon}}\right) \ \ x \in (0, 1),$$

when we transform it back to the original variable $x/\sqrt{\varepsilon}$. Also, in the other half of the domain we have the bound as

$$\left|\left|\frac{\partial^{i+k} w_l}{\partial x^i \partial t^k}\right|\right| \le C\left(1 + \exp\left(\frac{-x}{\sqrt{\varepsilon}}\right)\right), \ \ x \in (0, 1).$$

The proof of the bound on the right layer function w_r can be obtained analogously to that of the left layer function. Collecting the individual bounds together, we obtain

$$||u^{(i,k)}(x,t)|| \ \ \le \ \ C\left[1 + \varepsilon^{-\frac{i}{2}}\left(\exp(-x/\sqrt{\varepsilon}) + \exp(-(1-x)/\sqrt{\varepsilon})\right)\right]. \qquad (2.1.5)$$

Next we discretize problem (2.1.1)-(2.1.2) in space with the FMFDM.

2.2 Spatial discretization with the FMFDM

In this section we explore a FMFDM to discretize the spatial variable in problem (2.1.1)-(2.1.2). Problems of type (2.1.1)-(2.1.2) are known to be characterised by two boundary layers, thus we employ the transition parameter σ, defined as

$$\sigma = \min\left\{\frac{1}{4}, \sigma_0\sqrt{\varepsilon}\ln n\right\},$$

and the mesh width

$$\Delta x_i = x_i - x_{i-1} := \begin{cases} 4\sigma n^{-1}, & i = 1, 2, ..., n/4, \\ 2(1-2\sigma)n^{-1}, & i = n/4+1, ..., 3n/4, \\ 4\sigma n^{-1} & i = 3n/4+1, ..., n, \end{cases}$$

which leads to the partition

$$\{0 = x_0 < .. < \sigma < ... < 1 - \sigma < ... < x_n = 1\}.$$

Here σ_0 is a constant and n is the number of sub-intervals.

2.2.1 The FMFDM

We denote the approximation of $u(x_i, t)$ by $U_i(t)$ and discretize problem (2.1.1)-(2.1.2) in space to obtain the semi-discrete ordinary differential equations (ODEs)

$$\begin{aligned}\mathcal{L}^n U_i(t) \;\equiv\; & \frac{dU_i(t)}{dt} - \frac{2\varepsilon}{\Delta x_i + \Delta x_{i+1}}\left(\frac{U_{i+1}(t) - U_i(t)}{\Delta x_{i+1}} - \frac{U_i(t) - U_{i-1}(t)}{\Delta x_i}\right) + b_i(t)U_i(t) \\ & = f_i(t), \quad \text{for} \quad i = 1, 2, ...n-1, \end{aligned} \tag{2.2.6}$$

with the semi-discrete boundary and initial conditions

$$U_0(t) = \eta_0, \quad U_n(t) = \eta_1 \text{ and } U_i(0) = \varphi_i. \tag{2.2.7}$$

Incorporating the boundary conditions, we write the scheme (2.2.6)-(2.2.7) in matrix notation

$$U'(t) + A(t)U(t) = F(t). \tag{2.2.8}$$

Here $A(t) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$, is a tridiagonal matrix and $F(t),\ U(t)$ are in $\mathbb{R}^{n-1}$. The entries of $A(t)$ and $F(t)$ are given as:

$$\begin{aligned}
A_{ii}(t) &= \frac{2\varepsilon}{\Delta x_i + \Delta x_{i+1}}\left(\frac{1}{\Delta x_{i+1}} + \frac{1}{\Delta x_i}\right) + b_i(t), && i = 1, 2, ..., n-1,\\
A_{i,i+1}(t) &= -\frac{2\varepsilon}{\Delta x_{i+1}(\Delta x_i + \Delta x_{i+1})}, && i = 1, 2, ..., n-2,\\
A_{i,i-1}(t) &= -\frac{2\varepsilon}{\Delta x_i(\Delta x_i + \Delta x_{i+1})}, && i = 2, 3, ..., n-1,\\
F_1(t) &= f_1(t) + \left(\frac{2\varepsilon}{\Delta x_1(\Delta x_1 + \Delta x_2)}\right)\eta_0, &&\\
F_i(t) &= f_i(t), && i = 2, 3, ..., n-2,\\
F_{n-1}(t) &= f_{n-1}(t) + \left(\frac{2\varepsilon}{\Delta x_n(\Delta x_{n-1} + \Delta x_n)}\right)\eta_1. &&
\end{aligned}$$

It is to be noted that the difference scheme considered here produces a coefficient matrix $A(t)$ which is positive definite, thus it is an M-matrix [55] and hence the solution of equation (2.2.8) exist and is unique. Next we analyse the scheme (2.2.6)-(2.2.7) for convergence. .

2.2.2 Error analysis

Before we proceed with the error analysis, we highlight some properties of the semi-discrete problem (2.2.6)-(2.2.7) which play a major role in the said analysis. These properties have been established from [12].

Lemma 2.2.1. *(Semi-discrete maximum principle). Let* $\xi_i(t)$ *be a semi-discrete function defined on* $\bar{\Omega}^n \times [0,T]$ *and satisfies* $\xi_0(t) \geq 0,\ \ \xi_n(t) \geq 0,\ \ \forall\ 0 \leq i \leq n,\ \ t \in [0,T]$. *Then* $\mathcal{L}^n \xi_i(t) > 0, \forall\, (x_i,t) \in\ \Omega^n \times [0,T]$ *implies* $\xi_i(t) \geq 0, \forall\, (x_i,t) \in\ \bar{\Omega}^n \times [0,T]$.

Proof. Let l be an index such that

$$\xi_l(t) = \min_{x_i \in \bar{\Omega}^n;\, t \in [0,T]} \xi_i(t),$$

holds and assume $\xi_l(t) < 0$. Clearly $l \neq 0,\ \ l \neq n$. We have $(\xi_l(t))_t = 0,\ \ \xi_{l+1}(t) - \xi_l(t) \geq 0,$ and $\xi_l(t) - \xi_{l-1}(t) \geq 0$. It follows that

$$\begin{aligned}
\mathcal{L}^n \xi_i(t) &= (\xi_l(t))_t - \varepsilon \delta^2 \xi_l(t) + b_l(t)\xi_l(t) \\
&= (\xi_l(t))_t - \frac{2\varepsilon}{h_{i+1}+h_i}\left[\frac{\xi_{l+1}(t)-\xi_l(t)}{h_{i+1}} - \frac{\xi_{l+1}(t)-\xi_l(t)}{h_i}\right] + b_l(t)\xi_l(t) \\
&\leq 0,
\end{aligned}$$

which is a contradiction. Therefore, $\xi_i(t) \geq 0,\ \forall\ (x_i,t) \in \bar{\Omega}^n \times [0,T]$. □

Lemma 2.2.2. *(Uniform stability estimate). If* $u_i(t)$ *is the solution of the semi-discrete problem (2.2.6)-(2.2.7), then it satisfies the bound*

$$|u_i(t)| \leq \beta^{-1} \max_{(x_i,t)\in\bar{\Omega}^n\times[0,T]} |\mathcal{L}^n u_i(t)| + \max_{(x_i,t)\in\bar{\Omega}^n\times[0,T]} (|\varphi_i|, \max(\eta_0,\eta_1)).$$

Proof. We consider the functions $\Psi_i^{\pm}(t)$ defined by

$$\Psi_i^{\pm}(t) = p \pm u_i(t),$$

where we have used the definition

$$p = \beta^{-1} \max_{(x_i,t)\in\bar{\Omega}^n\times[0,T]} |\mathcal{L}^n u_i(t)| + \max_{(x_i,t)\in\bar{\Omega}^n\times[0,T]} (|\varphi_i|, \max(\eta_0,\eta_1)).$$

At the boundaries we have

$$\Psi_0^{\pm}(t) = p \pm u_0(t) = p \pm \eta_0 \geq 0,$$

$$\Psi_n^{\pm}(t) = p \pm u_n(t) = p \pm \eta_1 \geq 0.$$

Further on the domain $0 < i < n$ we obtain

$$\begin{aligned}
\mathcal{L}^n\Psi_i^{\pm}(t) &= -\frac{2\varepsilon}{\Delta x_i + \Delta x_{i+1}}\left(\frac{p \pm u_{i+1}(t) - p \pm u_i(t)}{\Delta x_{i+1}} - \frac{p \pm u_i(t) - p \pm u_{i-1}(t)}{\Delta x_i}\right) \\
&\quad + b_i(t)(p \pm u_i(t)) + (p \pm u_i(t))_t \\
&= b_i(t)p \pm \mathcal{L}^n u_i(t) \\
&= b_i(t)\left[\beta^{-1}\max_{(x_i,t)\in\Omega^n\times[0,T]}|\mathcal{L}^n u_i(t)| + \max_{(x_i,t)\in\Omega^n\times[0,T]}(|\varphi_i|, \max(\eta_0,\eta_1))\right] \pm f_i(t) \\
&\geq 0, \quad b_i(t) \geq \beta,
\end{aligned}$$

Therefore, from the semi-discrete maximum principle 2.2.1, $\Psi_i^{\pm}(t) \geq 0, \ \forall\ (x_i,t) \in \bar{\Omega}^n \times [0,T]$. This ends the proof. □

Now we decompose the numerical solution $U_i(t)$ into its regular and singular components

$$U_i(t) = V_i(t) + W_i(t).$$

This decomposition is done in order to obtain the ε-uniform bound of the error. Each term satisfies the differential equation

$$\mathcal{L}^n V_i(t) = f_i(t), \text{ on } \Omega^n \times (0,T], \ \ V_i(t) = v_i(t), \text{ on } \partial\Omega^n \times (0,T],$$
$$\mathcal{L}^n W_i(t) = 0, \text{ on } \Omega^n \times (0,T], \ \ W_i(t) = -v_i(t), \text{ on } \partial\Omega^n \times (0,T].$$

Now we write the error as

$$|U_i(t) - u_i(t)| \leq |V_i(t) - v_i(t)| + |W_i(t) - w_i(t)|,$$

and estimate each term separately.

Error of the regular component

We define the truncation error of the smooth component as

$$\begin{aligned}
\mathcal{L}^n(V_i(t) - v_i(t)) &= f_i(t) - \mathcal{L}^n v_i(t), \\
&= (\mathcal{L} - \mathcal{L}^n)v_i(t), \\
&= -\varepsilon\left(\frac{\partial^2}{\partial x^2} - \delta^2\right)v_i(t).
\end{aligned}$$

This does not include the time derivative term thus we estimate the error as though the problem is a stationary problem. Following [34] we present the theoretical error analysis. To achieve this goal we consider two cases; one is when x_i lies in the layer region and the other is when it lies outside the layer region. If $x_i \in [\sigma, 1-\sigma]$ then we have the error estimate as

$$|\mathcal{L}^n(V_i(t)-v_i(t))| = \left|-\varepsilon\left[(v_{xx}(t))_i - \frac{2}{\Delta x_i + \Delta x_{i+1}}\left(\frac{v_{i+1}(t)-v_i(t)}{\Delta x_{i+1}} - \frac{v_i(t)-v_{i-1}(t)}{\Delta x_i}\right)\right]\right|.$$

A truncated Taylor series expansions of $v_{i+1}(t)$ and $v_{i-1}(t)$ using the integral remainder terms gives

$$\begin{aligned}\frac{v_{i+1}(t)-v_i(t)}{\Delta x_{i+1}} - \frac{v_i(t)-v_{i-1}(t)}{\Delta x_i} &= \frac{\Delta x_{i+1}}{2}(v_{xx}(t))_i + \frac{\Delta x_i}{2}(v_{xx}(t))_i \\ &+\frac{1}{2!\Delta x_{i+1}}\int_{x_i}^{x_{i+1}}(x_{i+1}-s)^2(v_{xxx}(t))_i ds \\ &+\frac{1}{2!\Delta x_i}\int_{x_i}^{x_{i-1}}(x_{i-1}-s)^2(v_{xxx}(t))_i ds.\end{aligned}$$

Thus we have the error as

$$\begin{aligned}|\mathcal{L}^n(V_i(t)-v_i(t))| &= \left|-\varepsilon\left[v_{xx}(t))_i - \frac{2}{\Delta x_i + \Delta x_{i+1}}\left(\frac{\Delta x_{i+1}}{2}(v_{xx}(t))_i + \frac{\Delta x_i}{2}(v_{xx}(t))_i\right.\right.\right. \\ &\left.\left.\left.+\frac{1}{2!\Delta x_{i+1}}\int_{x_i}^{x_{i+1}}(x_{i+1}-s)^2(v_{xxx}(t))_i ds +\frac{1}{2!\Delta x_i}\int_{x_i}^{x_{i-1}}(x_{i-1}-s)^2(v_{xxx}(t))_i ds\right)\right]\right| \\ &= \left|\frac{\varepsilon}{(\Delta x_i+\Delta x_{i+1})}\left(\frac{1}{\Delta x_{i+1}}\int_{x_i}^{x_{i+1}}(x_{i+1}-s)^2(v_{xxx}(t))_i ds + \frac{1}{\Delta x_i}\int_{x_i}^{x_{i-1}}(x_{i-1}-s)^2(v_{xxx}(t))_i ds\right)\right| \\ &\leq \varepsilon\frac{|(v_{xxx}(t))_i|}{(\Delta x_i+\Delta x_{i+1})}\left[\frac{1}{\Delta x_{i+1}}\int_{x_i}^{x_{i+1}}(x_{i+1}-s)^2 ds - \frac{1}{\Delta x_i}\int_{x_{i-1}}^{x_i}(s-x_{i-1})^2 ds\right] \\ &\leq \varepsilon\frac{|(v_{xxx}(t))_i|}{(\Delta x_i+\Delta x_{i+1})}\left[\frac{1}{3\Delta x_{i+1}}(x_{i+1}-x_i)^3 - \frac{1}{3\Delta x_i}(x_i-x_{i-1})^3\right]\end{aligned}$$

$$\leq \frac{\varepsilon|(v_{xxx}(t))_i|}{3(\Delta x_i + \Delta x_{i+1})}\left[(x_{i+1}-x_i)^2 - (x_i - x_{i-1})^2\right]$$

$$\leq \frac{\varepsilon}{3}(x_{i+1}-x_{i-1})|(v_{xxx}(t))_i|.$$

Note that the mesh is very fine in the layer region, thus we bound the error as

$$|\mathcal{L}^n(V_i(t)-v_i(t))| = \left|-\varepsilon\left[(v_{xx}(t))_i - \frac{v_{i+1}(t)-2v_i(t)+v_{i-1}(t)}{\Delta x_i^2}\right]\right|.$$

Again using appropriate Taylor series expansion yields

$$\begin{aligned}\frac{v_{i+1}(t)-2v_i(t)+v_{i-1}(t)}{\Delta x_i^2} &= (v_{xx}(t))_i + \frac{\Delta x_i^2}{12}(v_{xxxx}(t))_i \\ |\mathcal{L}^n(V_i(t)-v_i(t))| &= \left|-\varepsilon\left[(v_{xx}(t))_i - (v_{xx}(t))_i - \frac{\Delta x_i^2}{12}(v_{xxxx}(t))_i\right]\right| \\ &\leq \varepsilon\frac{\Delta x_i^2}{12}|(v_{xxxx}(t))_i|.\end{aligned}$$

Collecting these two results together, we obtain the error estimate of the regular component as

$$|\mathcal{L}^n V_i(t) - v_i(t)| \leq \begin{cases} C\varepsilon(x_{i+1}-x_{i-1})|(v_{xxx}(t))_i|, & x_i \in [\sigma, 1-\sigma], \\ C\varepsilon(x_i - x_{i-1})^2|(v_{xxxx}(t))_i|, & \text{otherwise.}\end{cases}$$

Using the bounds $(v_{xxx}(t))_i$, $(v_{xxxx}(t))_i$ and $x_i - x_{i-1} < Cn^{-1}$ gives

$$|\mathcal{L}^n(V_i(t)-v_i(t))| \leq \begin{cases} C\sqrt{\varepsilon}n^{-1}, & x_i \in [\sigma, 1-\sigma], \\ Cn^{-2}, & \text{otherwise.}\end{cases}$$

The choice of transition parameter indicates a second order method, however, the results of the truncation error says otherwise. Thus we introduce the comparison function

$$\Phi(x_i,t) = C(n^{-2} + n^{-2}\frac{\sigma}{\sqrt{\varepsilon}}\theta(x_i)), \quad \text{where } \theta \text{ is a piecewise linear polynomial defined by}$$

$$\theta(x_i) = \begin{cases} x\sigma^{-1}, & 0 \leq x_i \leq \sigma, \\ 1, & \sigma \leq x_i \leq 1-\sigma, \\ (1-x_i)\sigma^{-1}, & 1-\sigma \leq x_i \leq 1, \end{cases} \qquad \delta\theta(x_i) = \begin{cases} -n\sigma^{-1}, & x \in (\sigma, 1-\sigma), \\ 0, & \text{otherwise.}\end{cases}$$

From the value of the transition parameter we have that for all $x_i \in \Omega^n$, and $t \in (0,T]$, the barrier function yields

$$\Phi(x_i,t) = Cn^{-2} + cn^{-2}\frac{\sigma}{\sqrt{\varepsilon}},$$

$$\leq \quad Cn^{-2} + Cn^{-2}(\sigma_0 \ln n),$$
$$\leq \quad Cn^{-2} \ln n.$$

When $x_i \in [\sigma, 1-\sigma]$, the semi-discrete operator on the barrier function gives

$$\begin{aligned}
\mathcal{L}^n \Phi(x_i,t) &= \Phi_t(x_i,t) - \varepsilon\delta^2\Phi(x_i,t) + b(x_i,t)\Phi(x_i,t) \\
&= -\varepsilon\delta\Phi(x_i,t) + b(x_i,t)\Phi(x_i,t) \\
&= \sqrt{\varepsilon} b_i(t) n^{-1} + Cn^{-2} \geq C(\sqrt{\varepsilon} n^{-1} + n^{-2}),
\end{aligned}$$

and when x_i lies in the two layer regions we have

$$\mathcal{L}^n \Phi(x_i,t) \quad \geq \quad Cn^{-2}.$$

Combing these two estimates gives

$$\mathcal{L}^n \Phi(x_i,t) \geq \begin{cases} C\sqrt{\varepsilon} n^{-1} + cn^{-2}, & x_i \in [\sigma, 1-\sigma], \\ Cn^{-2}, & \text{otherwise.} \end{cases}$$

Now we introduce the functions

$$\Psi_i^{\pm}(t) = \Phi(x_i,t) \pm V_i(t) - v_i(t).$$

From Lemma 2.2.1 $\Psi_i^{\pm}(t) \geq 0, \ \forall\, (x_i,t) \in \partial\Omega^n \times [0,T]$, thus at each point $(x_i,t) \in \Omega^n \times [0,T]$, $\mathcal{L}^n \Psi_i^{\pm}(t) \geq 0$. It follows that

$$\Psi^{\pm}(x_i,t) \geq 0, \ \forall\ (x_i,t) \in \bar{\Omega}^n \times [0,T],$$

and hence $|V_i(t) - v_i(t)| \leq \Phi(x_i,t) \leq Cn^{-2} \ln n$. Which implies

$$|V_i(t) - v_i(t)| \leq Cn^{-2} \ln n.$$

Error of the singular component

To estimate the error corresponding to the singular part, we write $W_i(t)$ analogously to the singular component of the exact solution as $W_i(t) = W_{ri}(t) + W_{li}(t)$, where each term also satisfies

$$\begin{aligned}
\mathcal{L}^n W_l(t) &= 0, \ (x_i,t) \in \Omega^n \times (0,T], \ W_l = -v^0, \ x_i \in \Omega^n, \ t \in (0,T], \\
& \quad W_l = 0, \quad t \in (0,T], \\
\mathcal{L}^n W_r(t) &= 0, \ \in (x_i,t) \in \Omega^n \times (0,T], \ W_r = -v^0, \ x_i \in \Omega^n, \ t \in (0,T], \\
& \quad W_r = 0, \ t \in (0,T].
\end{aligned}$$

Now the error is of the form

$$|W_i(t) - w_i(t)| = |W_l(x_i, t) - w_l(x_i, t)| + |W_r(x_i, t) - w_r(x_i, t)|.$$

To bound the error of the left layer function, there are two possibilities depending on the value of the transition parameter σ, that is, either $\sigma = \frac{1}{4}$ or $\sigma = \sigma_0\sqrt{\varepsilon}\ln n$. In the former case the mesh is uniform and thus the classical error estimate used for standard finite difference methods can be used. In the latter case the mesh is piecewise uniform and so the estimate will depend on the mesh spacing. Here we consider only the latter case because the difference scheme considered here is on a piecewise uniform mesh. Depending on the mesh spacing, we have two distinct error cases which are; the error in sub-intervals $[0, \sigma]$ and $[1-\sigma, 1]$ with the spacing $4\sigma/n$ and the spacing $2(1-2\sigma)/n$ for the sub-interval $[\sigma, 1-\sigma]$.

Note that in the sub-intervals $(0, \sigma)$ and $(1-\sigma, 1)$ the mesh is very fine so we bound the error as follows

$$|\mathcal{L}^n(W_l(t) - w_l(t))(x_i)| \quad = \quad -\varepsilon\left(\frac{\partial^2}{\partial x^2} - \delta^2\right) w_l(t).$$

Using appropriate Taylor series expansions gives the estimate

$$|\mathcal{L}^n(W_l(t) - w_l(t))(x_i)| \leq C\varepsilon(x_i - x_{i-1})^2|(w_{lxxxx}(t))_i|$$

From the bounds on the fourth derivative and the step size $\Delta x_i = 4\sigma n^{-1}$, we have

$$\begin{aligned}|\mathcal{L}^n(W_l(t) - w_l(t))(x_i)| &\leq C\varepsilon\left(\frac{4\sigma}{n}\right)^2 \varepsilon^{-2}\exp\left(\frac{-x}{\sqrt{\varepsilon}}\right)\\ &\leq Cn^{-2}\frac{\sigma^2}{\varepsilon}\exp\left(-\frac{x}{\sqrt{\varepsilon}}\right)\end{aligned}$$

However, $\sigma = \sigma_0\sqrt{\varepsilon}\ln n$ and hence the error becomes

$$\begin{aligned}|\mathcal{L}^n(W_l(t) - w_l(t))(x_i)| &\leq Cn^{-2}\frac{(\sigma_0\sqrt{\varepsilon}\ln n)^2}{\varepsilon}\exp\left(-\frac{x}{\sqrt{\varepsilon}}\right)\\ &\leq Cn^{-2}\ln n^2\exp\left(-\frac{x}{\sqrt{\varepsilon}}\right)\\ &\leq Cn^{-2}\ln n^2.\end{aligned}$$

When $x_i \in [\sigma, 1-\sigma)$ the error is given as

$$|\mathcal{L}^n(W_l(t) - w_l(t))(x_i)| = -\varepsilon|(w_{lxx}(t) - \delta^2 w_l(t))(x_i)|.$$

But $|\delta^2(w_l(t))(x_i)| \leq \max_{x_{i-1},x_{i+1}} |w_{lxx}(t)(x_i)|$, therefore we have

$$|\mathcal{L}^n(W_l(t) - w_l(t))(x_i)| \leq 2\varepsilon \max_{x_{i-1},x_{i+1}} |(w_{lxx}(t))(x_i)|.$$

Now using the step size $\frac{4\sigma}{n}$ we have $-x_{i-1} = -\sigma + \frac{4\sigma}{n}$ and so

$$\begin{aligned}
|\mathcal{L}^n(W_l(t) - w_l(t))(x_i)| &\leq C \exp\left(\frac{-x_{i-1}}{\sqrt{\varepsilon}}\right) \\
&\leq C \exp\left(\frac{-\sigma}{\sqrt{\varepsilon}}\right) \exp\left(\frac{4\sigma n^{-1}}{\sqrt{\varepsilon}}\right) \\
&\leq C \exp\left(-2\ln n\right) \exp\left(8n^{-1}\ln n\right) \\
&\leq Cn^{-2}(n^{\frac{1}{n}})^8 \\
&\leq Cn^{-2}.
\end{aligned}$$

Adding the results in each domain gives

$$\begin{aligned}
|\mathcal{L}^n(W_l(t) - w_l(t))(x_i)| &\leq Cn^{-2}\ln n^2 + Cn^{-2}, \\
&\leq Cn^{-2}\ln n^2,
\end{aligned}$$

for all $(x_i, t) \in \bar{\Omega}^n \times [0, T]$ which results in

$$|(W_l(t) - w_l(t))(x_i)| \leq C(n^{-1}\ln n)^2$$

on the application of Lemma 2.2.2. The estimate for the right layer function can be obtained in a similar manner and it is given as

$$|(W_r(t) - w_r(t))(x_i)| \leq C(n^{-1}\ln n)^2.$$

Lemma 2.2.3. *Let $u_i(t)$ be the exact solution of (2.1.1)-(2.1.2) and $U_i(t)$ the solution of (2.2.6)-(2.2.7) at $x = x_i$. Then we have*

$$\max_{0<\varepsilon\leq 1} \max_{0\leq i\leq n} |U_i(t) - u_i(t)| \leq C(n^{-1}\ln n)^2,$$

where C is a constant independent of n and ε.

Next, we design a fitted operator finite difference method to discretize in space.

2.3 Spatial discretization with the FOFDM

Now, we develop a FOFDM to discretize problem (2.1.1)-(2.1.2)in space. Then, we will analyse the method for convergence.

2.3.1 The FOFDM

Let n be a positive integer and divide the interval $[0,1]$ into the uniform sub-intervals

$$x_0 = 0,\ x_i = x_0 + i\Delta x,\ \ i = 1(1)n-1,\ \ \Delta x = x_i - x_{i-1},\ x_n = 1.$$

Denoting the approximation $u(x_i,t)$ by $U_i(t)$, we discretize the space variable to obtain the scheme

$$\mathcal{L}^{\Delta x}U_i(t) \equiv \frac{dU_i(t)}{dt} - \varepsilon\frac{U_{i+1}(t) - 2U_i(t) - U_{i-1}(t)}{\phi_i^2(\varepsilon,\Delta x,t)} + b_i(t)U_i(t) = f_i(t), \quad (2.3.9)$$

together with the semi-discrete boundary and initial conditions

$$U_0(t) = \eta_0,\ \ U_n(t) = \eta_1 \text{ and } U_i(0) = \varphi_i,\ \ i = 1,...,n-1. \quad (2.3.10)$$

Here the denominator function ϕ is given by

$$\phi_i(\varepsilon,\Delta x,t) = \frac{2}{\rho_i^2}\sinh\left(\frac{\rho_i\Delta x}{2}\right),$$

where $\rho_i = \sqrt{b_i(t)/\varepsilon}$. Similar to the previous section, we write the difference scheme (2.3.9)-(2.3.10) as

$$U'(t) + A(t)U(t) = F(t), \quad (2.3.11)$$

where $U(t), F(t) \in \mathbb{R}^{n-1}$ and $A(t) \in \mathbb{R}^{n-1}\times\mathbb{R}^{n-1}$. The entries of $A(t)$ and $F(t)$ are given by

$$\begin{aligned}
A_{ii}(t) &= \frac{2\varepsilon}{\phi_i^2(\varepsilon,\Delta x,t)} + b_i(t), && i = 1,2,...,n-1,\\
A_{i,i+1}(t) &= -\frac{\varepsilon}{\phi_i^2(\varepsilon,\Delta x,t)}, && i = 1,2,...,n-2,\\
A_{i,i-1}(t) &= -\frac{\varepsilon}{\phi_i^2(\varepsilon,\Delta x,t)}, && i = 2,3,...,n-1,\\
F_1(t) &= f_1(t) + \frac{\varepsilon}{\phi_1^2(\varepsilon,\Delta x,t)}\eta_0, &&\\
F_i(t) &= f_i(t), && i = 2,3,...,n-2,\\
F_{n-1}(t) &= f_{n-1}(t) + \frac{\varepsilon}{\phi_{n-1}^2(\varepsilon,\Delta x,t)}\eta_1. &&
\end{aligned}$$

Again, the coefficient matrix $A(t)$ is also an M-matrix and thus the semi-discrete problem (2.3.9)-(2.3.10) admits the maximum principle which also ensures stability of the solution. Note that these properties have been stated in Lemmas 2.2.1 and 2.2.2 so we will refer to them when the needed arises to avoid repetition.

Now, we analyse the spatial discretization for convergence.

2.3.2 Error analysis

The following lemma will be needed to prove the uniform convergence of the numerical method (2.3.9)-(2.3.10) [41].

Lemma 2.3.1. *For all integers j on a fixed mesh , we have that*

$$\lim_{\varepsilon\to} \max_{1<i<n-1} \frac{\exp(-Cx_i\sqrt{\varepsilon})}{\varepsilon^{j/2}} = 0$$

and

$$\lim_{\varepsilon\to} \max_{1<i<n-1} \frac{\exp(-C(1-x_i)/\sqrt{\varepsilon})}{\varepsilon^{j/2}} = 0,$$

where $x_i = i\Delta x,\ \ \Delta x = 1/n,\ \ \forall \quad i = 1(1)n-1.$

Proof. When the domain $[0,1]$ is converted in to the discrete domain $[0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1]$, we see that the interior grid point satisfy the inequalities

$$\max_{1<i<n-1} \frac{\exp(-Cx_i/\sqrt{\varepsilon})}{\varepsilon^{j/2}} \leq \frac{\exp(-Cx_1/\sqrt{\varepsilon})}{\varepsilon^{j/2}} = \frac{\exp(-C\Delta x/\sqrt{\varepsilon})}{\varepsilon^{j/2}},$$

and

$$\max_{1\leq i\leq n-1} \frac{\exp(-C(1-x_i)/\sqrt{\varepsilon})}{\varepsilon^{j/2}} \leq \frac{\exp(-C(1-x_n)/\sqrt{\varepsilon})}{\varepsilon^{j/2}} = C\frac{\exp(-C\Delta x/\varepsilon)}{\varepsilon^{j/2}}$$

Since $x_1 = \Delta x,\ \ 1-x_{n-1} = 1-(n-1)\Delta x = 1-n\Delta x+\Delta x = \Delta x$. Applying L'Hospitals's rule results in

$$\lim_{\varepsilon\to 0} \frac{\exp(-C\Delta x/\sqrt{\varepsilon})}{\varepsilon^{j/2}} = \lim_{p=1/\sqrt{\varepsilon}\to\infty} \frac{p^j}{\exp(C\Delta xp)} \equiv \lim_{p\to\infty} \frac{j!}{(C\Delta x)^j \exp(C\Delta xp)} = 0.$$

$\square$

Now we estimate the truncation error as follows:

$$\begin{aligned}\mathcal{L}^{\Delta x}(U_i(t)-u_i(t)) &= f_i(t)-\mathcal{L}^{\Delta x}u_i(t)\\ &= \left(\mathcal{L}-\mathcal{L}^{\Delta x}\right)u_i(t),\\ &= \varepsilon\left[-(u_{xx}(t))_i+\frac{u_{i+1}(t)-2u_i(t)+u_{i-1}(t)}{\phi_i^2(\varepsilon,\Delta x,t)}\right].\end{aligned}$$

Since the time derivative does not appear in the expression above, the analysis will be done as for a stationary problem. Using a truncated Taylor series expansion of $u_{i+1}(t)$ and $u_{i-1}(t)$ gives

$$\begin{aligned}\mathcal{L}^{\Delta x}(U_i(t)-u_i(t)) &= -\varepsilon(u_{xx}(t))_i+\frac{\varepsilon}{\phi_i^2}\left(\Delta x^2(u_{xx}(t))_i+\frac{\Delta x^4}{12}(u_{xxxx}(t))_i\xi\right)\\ & \quad \xi\in(u_{i+1},u_{i-1}).\end{aligned}$$

We expand the denominator function ϕ_i^{-2} in Taylor series to obtain $1/\Delta x^2-\rho_i^2/12$ $+\rho_i^4\Delta x^2/240$. The error yields

$$\begin{aligned}\mathcal{L}^{\Delta x}(U_i(t)-u_i(t)) &= \left(\frac{\varepsilon}{\Delta x^2}-\frac{\varepsilon\rho_i^2}{12}+\frac{\varepsilon\rho_i^4\Delta x^2}{240}\right)\left(\Delta x^2(u_{xx}(t))_i+\frac{\Delta x^4}{12}(u_{xx}(t))_i\xi\right)\\ &\quad -\varepsilon(u_{xx}(t))_i,\quad \xi\in(u_{i+1},u_{i-1}),\\ &= -\varepsilon(u_{xx}(t))_i+\varepsilon(u_{xx}(t))_i+\varepsilon\frac{\Delta x^2}{12}(u_{xxxx}(t))_i\xi-\varepsilon\frac{\rho_i^2\Delta x^2}{12}(u_{xx}(t))_i-\\ &\quad \varepsilon\frac{\rho_i^2\Delta x^4}{144}(u_{xxxx}(t))_i\xi+\varepsilon\frac{\rho_i^4\Delta x^4}{240}(u_{xx}(t))_i+\varepsilon\frac{\rho_i^4\Delta x^6}{2880}(u_{xxxx}(t))_i\xi\\ &= \left(\frac{\varepsilon}{12}(u_{xxxx}(t))_i\xi-\varepsilon\frac{\rho_i^2}{12}(u_{xx}(t))_i\right)\Delta x^2-+\left(\varepsilon\frac{\rho_i^4}{2880}(u_{xxxx}(t))_i\xi\right)\Delta x^6\\ &\quad \left(\varepsilon\frac{\rho_i^2}{144}(u_{xxxx}(t))_i\xi-\varepsilon\frac{\rho_i^4}{240}(u_{xx}(t))_i\right)\Delta x^4\end{aligned}$$

Applying the bounds on the derivatives and Lemma 2.3.1 results in

$$\begin{aligned}|\mathcal{L}^{\Delta x}(U_i(t)-u_i(t))| &= \left(\frac{\varepsilon}{12}-\varepsilon\frac{\rho_i^2}{12}\right)\Delta x^2-\left(\varepsilon\frac{\rho_i^2}{144}-\varepsilon\frac{\rho_i^4}{240}\right)\Delta x^4+\left(\varepsilon\frac{\rho_i^4}{2880}\right)\Delta x^6\\ &\leq C\Delta x^2,\end{aligned}$$

where we have used the relation $\Delta x^2>\Delta x^4>...,$. Invoking Lemma 2.2.2 yields

$$|U_i(t)-u_i(t)|\leq C\Delta x^2.$$

Lemma 2.3.2. *The fitted operator finite difference scheme (2.3.9)-(2.3.10) satisfies the theoretical error*

$$\max_{0<\varepsilon\leq 1}\max_{0\leq i\leq n}|U_i(t)-u_i(t)|\leq C\Delta x^2,$$

where $u_i(t)$ is the exact solution of (2.1.1)-(2.1.2), $U_i(t)$ the numerical solution of (2.3.9)-(2.3.10) and C is a constant independent of Δx and ε.

2.4 Time discretization

In this section we integrate the IVPs which resulted from the schemes (2.2.6)-(2.2.7) and (2.3.9)-(2.3.10) using the backward Euler integration technique on a uniform mesh. Before we proceed, we suppose $u(t)\in\mathcal{C}^2((0,T])$. Throughout this thesis, we use τ and K to denote the mesh width and the number of sub-intervals. Also, we denote the approximation of $u_i(t_k) := u_i^k$ by U_i^k, however for notational simplicity, we drop the subscript index. Now we define τ by T/K, and thus write the fully discrete scheme as

$$\frac{U^k-U^{k-1}}{\tau}+A(t_k)U^k=F(t_k),\quad \text{for } k=1,...,K, \tag{2.4.12}$$

with initial condition $U(0)=\varphi$. Rearranging equation (2.4.12) we obtain

$$U^k=(\mathbf{I}+\tau A(t_k))^{-1}(\tau F(t_k)+U^{k-1}).$$

Now we estimate the error associated with this discretization. Let $u(t_k)$ be the exact solution at a time t_k and U^k be the numerical solution. Then local truncation error e^k for the time integration is given by

$$\begin{aligned} e^k &= u(t_k)-U^k \\ &= u(t_k)-[\mathbf{I}+\tau A(t_k)]^{-1}(\tau F(t_k)+u(t_{k-1}). \end{aligned}$$

Using a truncated Taylor series expansion of the term $u(t_{k-1})$

$$u(t_{k-1})\equiv u(t_k)-\tau u'(t_k)+\frac{\tau^2}{2}u''(t_k)-\frac{\tau^3}{3!}u'''(t_k)+\mathcal{O}\tau^4,$$

and $u'(t_k)=F(t)-A(t_k)u(t)$, gives

$$u(t_{k-1})=u(t_k)-\tau[F(t_k)-A(t_k)u(t_k)]+\frac{\tau^2}{2}u''(t_k)-\frac{\tau^3}{3!}u'''(t_k)+\mathcal{O}\tau^4.$$

The local truncation error e^k yields

$$\begin{aligned} e^k &= u(t_k) - [\mathbf{I} + \tau A(t_k)u]^{-1}\left[u(t_k) + \tau A(t_k)u(t_k) + \frac{\tau^2}{2}u''(t_k) - \frac{\tau^3}{3!}u'''(t_k) + \mathcal{O}\tau^4\right] \\ &= u(t_k) - [\mathbf{I} + \tau A(t_k)]^{-1}\left[[\mathbf{I} + \tau A(t_k)]u(t_k) + \frac{\tau^2}{2}u''(t_k) - \frac{\tau^3}{3!}u'''(t_k) + \mathcal{O}\tau^4\right]. \\ &= \quad [\mathbf{I} + \tau A(t_k)]^{-1}\left(\frac{\tau^2}{2}u''(t_k) - \frac{\tau^3}{3!}u'''(t_k) + \mathcal{O}\tau^4\right). \end{aligned}$$

From the absolute monotonicity condition 1.3.9

$$e^k = \frac{\tau^2}{2}u''(t_k) - \frac{\tau^3}{3!}u'''(t_k) + \mathcal{O}\tau^4,$$

follows. Now using the relation $\tau^3 < \tau^2$ for small τ, and $u(t) \leq C$, we obtain

$$||e^k|| \leq C\tau^2.$$

Lemma 2.4.1. *The local truncation error associated with the time integration satisfies*

$$||e^k|| \leq C\tau^2$$

where C is a constant independent of ε and K.

From here, the global error $\mathcal{E}^K$ is given by

$$\mathcal{E}^K \quad \leq \sum_{k=1}^{K} e^k \leq CK\tau^2 \leq C\tau. \tag{2.4.13}$$

Lemma 2.4.2. *The global error $\mathcal{E}^K$ of the time discretization, satisfies*

$$||\mathcal{E}^K|| \leq C\tau.$$

The main results in this chapter is summarized in Theorem 2.4.1 below.

Theorem 2.4.1. *Let $u \in \mathcal{C}^{4,2}(\bar{Q})$ be the exact solution of the continuous problem (2.1.1)-(2.1.2) and U_i^k be the numerical solution obtained via the FMDML (2.2.6-(2.2.7) along with (2.4.12) or the FOFDML (2.3.9)-(2.3.10) along with (2.4.12). Then the errors of these methods are as follows:*

$$\sup_{0<\varepsilon\leq 1} \max_{0\leq i\leq n; 0\leq k\leq K} ||U_i^k - u_i^k|| \leq C((n^{-1}\ln n)^2 + \tau), \quad \textit{for the FMFDML}$$

and

$$\sup_{0<\varepsilon\leq 1} \max_{0\leq i\leq n; 0\leq k\leq K} ||U_i^k - u_i^k|| \leq C(\Delta x^2 + \tau), \quad \textit{for the FOFDML.}$$

2.5 Numerical example

In this section, we verify the theoretical results on the test problem below. We compute the maximum pointwise error and the numerical rate of convergence for both the FMFDML and FOFDML. In computations involving the FMFDML, we use $\sigma_0 = 2$. The exact solution of the test problem is not known. Therefore, we use the double mesh principle to compute the maximum pointwise error as follows.

$$E^{\varepsilon}_{n,\tau} = \max_{0\leq i\leq n; 0\leq k\leq K} |U^{k;K}_{i;n} - U^{k;4K}_{i;2n}|, \tag{2.5.14}$$

where $U^{k;K}_{i;n}$ is the numerical solution and $U^{k;K}_{i;2n}$ is also a numerical solution but on the mesh $\mu(2n, 4K)$. We have used $2n$ and $4K$ in order to balance the error between the time and space variables in the fully discrete scheme. Also, we compute the rate of convergence using the formula

$$r_l = \log_2\left(E^{\varepsilon}_{n,\tau}/E^{\varepsilon}_{2n,\tau/4}\right), \quad l = 1, 2, ... \tag{2.5.15}$$

The error analysis shows a first order accuracy for the backward Euler, an almost second order accuracy for the FMFDML and a second order accuracy for the FOFDML, all with respect to the perturbation parameter. These results which are summarized in Theorem 2.4.1, are in conformity with the numerical results in Tables 2.1-2.4. Tables 2.1-2.2 display the results obtained for the maximum pointwise error for the FMFDML and the FOFDML. Likewise Tables 2.3-2.4 show the rate at which the numerical solution is converging to the exact solution.

Example 2.5.1. *[43] Consider the problem*

$$u_t - \varepsilon u_{xx} + \frac{1+x^2}{2}u = \exp(x) - 1 + \sin(\pi x) \;\; (x,t) \in (0,1)\times(0,1],$$
$$u(x,0) = 0, \quad x \in \bar{\Omega}, \;\; u(0,t) = u(1,t) = 0, \quad t \in (0,1].$$

Table 2.1: Maximum pointwise error for Example 2.5.1 using the FMFDML

ε	$n = 32$	64	128	256	512
	$K = 10$	40	160	640	2560
10^{0}	$1.68E-02$	$5.74E-03$	$1.58E-03$	$4.04E-04$	$1.02E-04$
10^{-1}	$2.43E-02$	$6.48E-03$	$1.65E-03$	$4.14E-04$	$1.04E-04$
10^{-2}	$2.63E-02$	$6.86E-03$	$1.73E-03$	$4.35E-04$	$1.09E-04$
10^{-3}	$3.40E-02$	$1.08E-02$	$2.77E-03$	$6.98E-04$	$1.75E-04$
10^{-4}	$3.38E-02$	$1.13E-02$	$3.46E-03$	$1.04E-03$	$3.10E-04$
10^{-5}	$3.38E-02$	$1.13E-02$	$3.45E-03$	$1.04E-03$	$3.10E-04$
10^{-6}	$3.38E-02$	$1.13E-02$	$3.45E-03$	$1.04E-03$	$3.10E-04$
10^{-7}	$3.38E-02$	$1.13E-02$	$3.45E-03$	$1.04E-03$	$3.10E-04$
10^{-8}	$3.38E-02$	$1.13E-02$	$3.45E-03$	$1.04E-03$	$3.10E-04$
10^{-9}	$3.38E-02$	$1.13E-02$	$3.45E-03$	$1.04E-03$	$3.10E-04$
10^{-10}	$3.38E-02$	$1.13E-02$	$3.45E-03$	$1.04E-03$	$3.10E-04$

Table 2.2: Maximum pointwise error for Example 2.5.1 using the FOFDML

ε	$n = 32$	64	128	256	512
	$K = 10$	40	160	640	2560
10^{0}	$1.68E-02$	$5.74E-03$	$1.58E-03$	$4.04E-04$	$1.02E-04$
10^{-1}	$2.42E-02$	$6.47E-03$	$1.64E-03$	$4.13E-04$	$1.03E-04$
10^{-2}	$2.59E-02$	$6.75E-03$	$1.71E-03$	$4.28E-04$	$1.07E-04$
10^{-3}	$3.26E-02$	$9.49E-03$	$2.51E-03$	$6.37E-04$	$1.60E-04$
10^{-4}	$3.26E-02$	$2.40E-02$	$1.12E-02$	$3.55E-03$	$9.67E-04$
10^{-5}	$2.34E-02$	$7.94E-03$	$1.65E-02$	$1.66E-02$	$7.20E-03$
10^{-6}	$2.35E-02$	$6.07E-03$	$1.61E-03$	$5.50E-03$	$1.81E-02$
10^{-7}	$2.35E-02$	$6.07E-03$	$1.53E-03$	$3.83E-04$	$7.13E-04$
10^{-8}	$2.35E-02$	$6.07E-03$	$1.53E-03$	$3.83E-04$	$9.59E-05$
10^{-9}	$2.35E-02$	$6.07E-03$	$1.53E-03$	$3.83E-04$	$9.59E-05$
10^{-10}	$2.35E-02$	$6.07E-03$	$1.53E-03$	$3.83E-04$	$9.59E-05$

Table 2.3: Rate of convergence for Example 2.5.1 using the FMFDML

ε	r_1	r_2	r_3	r_4
10^0	1.55	1.86	1.96	1.99
10^{-1}	1.91	1.97	1.99	2.00
10^{-2}	1.94	1.98	2.00	2.00
10^{-3}	1.66	1.96	1.99	2.00
10^{-4}	1.59	1.70	1.73	1.75
10^{-5}	1.59	1.70	1.73	1.75
10^{-6}	1.59	1.70	1.73	1.75
10^{-7}	1.59	1.70	1.73	1.75
10^{-8}	1.59	1.70	1.73	1.75
10^{-9}	1.59	1.70	1.73	1.75
10^{-10}	1.59	1.70	1.73	1.75

Table 2.4: Rate of convergence for Example 2.5.1 using the FOFDML

ε	r_1	r_2	r_3	r_4
10^0	1.55	1.86	1.96	1.99
10^{-1}	1.91	1.97	1.99	2.00
10^{-2}	1.94	1.98	2.00	2.00
10^{-3}	1.78	1.92	1.98	1.99
10^{-4}	0.44	1.09	1.66	1.87
10^{-5}	1.56	-1.05	-0.01	1.20
10^{-6}	1.95	1.92	-1.78	-1.72
10^{-7}	1.95	1.99	2.00	-0.89
10^{-8}	1.95	1.99	2.00	2.00
10^{-9}	1.95	1.99	2.00	2.00
10^{-10}	1.95	1.99	2.00	2.00

2.6 Conclusion

In this chapter, we considered one-dimensional reaction diffusion problems. We started by presenting qualitative properties pertaining to these problems. Then we reviewed a special case of the fitted mesh finite difference method of lines (FMFDML) of [12]. This FMFDML is second order convergent in space (except for a logarithmic factor) and first order convergent in time. Further we designed a fitted operator finite difference method of lines (FOFDML). This method consists of a space discretization via fitted operator finite difference method followed by a time discretization using a backward Euler method. Convergence analysis shows that this FOFDML is second order accurate uniformly with respect to the perturbation parameter ε. To illustrate this method in practice, we conducted numerical simulations on a test example. The computed results confirmed our theoretical results.

Chapter 3

Methods of Lines for One-Dimensional Convection-Diffusion Problems

In this chapter, devoted to one-dimensional convection-diffusion problems, we first explore a particular case of the FMFDML in [12]. Then, we develop a FOFDML. After presenting some qualitative results relating to the convection-diffusion problems under study, we present the methods, their convergence analysis and some numerical results to illustrate the performance of the algorithms.

3.1 Continuous problem

We consider the problem

$$\mathcal{L}u(x,t) \equiv u_t - \varepsilon u_{xx} + a(x,t)u_x + b(x,t)u = f(x,t),\ (x,t) \in Q = \Omega \times (0,T],\ \ \Omega = (0,1). \tag{3.1.1}$$

Subject to the boundary and the initial condition

$$u(0,t) = \eta_0, \quad u(1,t) = \eta_1,\ \ u(x,0) = \varphi(x),\ \ x \in \bar{\Omega},\ t \in [0,T]. \tag{3.1.2}$$

The perturbation parameter ε is such that $0 < \varepsilon << 1$, the coefficient functions $a(x,t),\ b(x,t)$ and $f(x,t)$ are sufficiently smooth and satisfy

$$a(x,t) \geq \alpha > 0,\ \ b(x,t) \geq \beta > 0,\ \ \forall\ \ (x,t) \in \bar{Q}.$$

We assume enough compatibility conditions so that the solution will match at the corners of the domain. It is well known that the exact solution of problem (3.1.1)-(3.1.2) is characterised with layers at the neighbourhood $x = 1$, of Q. Similar to problem (2.1.1)-(2.1.2) in Chapter 2, problem (3.1.1)-(3.1.2) admits the continuous maximum principle as well as the uniform stability estimate in Lemmas 2.1.1 and 2.1.2 respectively.

Under the hypothesis of these two lemmas, the exact solution and its derivatives satisfy

$$\left\|\frac{\partial^{i+k}u(x,t)}{\partial x^i \partial t^k}\right\| \ \leq \ C\left(1+\varepsilon^{-i}\exp\left(\frac{-\alpha(1-x)}{\varepsilon}\right)\right),\ \forall\ (x,t) \in \bar{Q}. \qquad (3.1.3)$$

Where i and k are positive integers such that $0 \leq i \leq 3$ and $0 \leq i+j \leq 3$, [21]. Also, it admits the decomposition

$$u(x,t) = v(x,t) + w(x,t),$$

which represents the regular and singular components respectively. The regular component is the solution to the non-homogeneous problem

$$\mathcal{L}v(x,t) = f(x,t),\ (x,t) \in Q,\ v(0,t) = u(0,t),\ t \in [0,T],\ \ v(x,0) = \varphi,\ x \in \bar{\Omega},$$

and the layer component is the solution to the homogeneous problem

$$\begin{aligned} \mathcal{L}w(x,t) \ &= \ 0,\ (x,t) \in Q,\ w(x,0) = 0,\ x \in \Omega,\ w(0,t) = 0,\ t \in\ [0,T], \\ & w(1,t) = u(1,t) - v(1,t),\ t \in [0,T]. \end{aligned}$$

Further, the regular component can be written in the form

$$v(x,t) = v^0(x,t) + \varepsilon v^1(x,t) + \varepsilon^2 v^2(x,t),\ (x,t)\ \in \bar{Q}.$$

Where v_0 is the solution to the reduced problem

$$\begin{aligned} \mathcal{L}v^0 \ &\equiv \ v_t^0(x,t) + a(x,t)v_x^0(x,t) + b(x,t)v^0(x,t) = f(x,t),\ (x,t) \in Q \\ & v^0(x,0) = \varphi,\ \ x \in \Omega,\ v^0(0,t) = 0,\ \ t \in [0,T], \end{aligned}$$

and v^1, v^2 are the respective solutions to the problems

$$\begin{aligned}
\mathcal{L}v^1 &\equiv v_t^1(x,t) + a(x,t)v_x^1(x,t) + b(x,t)v^1(x,t) = v_{xx}^0(x,t),\ (x,t) \in Q,\\
&\quad v^1(x,0) = 0,\ \ x \in \bar{\Omega},\ \ v_1(0,t) = 0,\ \ t \in [0,T],\\
\mathcal{L}v^2 &\equiv v_t^2(x,t) - \varepsilon v_{xx}^2(x,t) + a(x,t)v_x^2(x,t) + b(x,t)v^2(x,t) = v_{xx}^1(x,t),\ (x,t) \in Q,\\
&\quad v^2(x,t) = 0,\ \ (x,t) \in \bar{Q}.
\end{aligned}$$

The functions v^j, $j = 0, 1$, are solutions to a problem which is independent of ε, hence they satisfy the bound

$$\left|\left|\frac{\partial^{i+k} v^j}{\partial x^i \partial t^k}\right|\right| \leq C.$$

For v^2 it is the solution to a problem similar to the original problem therefore it satisfies

$$\left|\left|\frac{\partial^{i+k} v^2}{\partial x^i \partial t^k}\right|\right| \leq C\left(1 + \varepsilon^{-i}\exp(\frac{-\alpha(1-x)}{\varepsilon}\right).$$

When we add these two results, we obtain

$$\begin{aligned}
\left|\left|\frac{\partial^{i+k} v}{\partial x^i \partial^k}\right|\right| &\leq \left|\left|\frac{\partial^{i+k} v^0}{\partial x^i \partial^k}\right|\right| + \varepsilon\left|\left|\frac{\partial^{i+k} v^1}{\partial x^i \partial^k}\right|\right| + \varepsilon^2\left|\left|\frac{\partial^{i+k} v^2}{\partial x^i \partial^k}\right|\right|\\
&\leq C + C\varepsilon + C\varepsilon^2\left(1 + \varepsilon^{-i}\exp(\frac{-\alpha(1-x)}{\varepsilon}\right)\\
&\leq C(1 + \varepsilon^{2-i}\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\\
&\leq C(1 + \varepsilon^{2-i}),\ \text{ since } e^{-\alpha(1-x)/\varepsilon} \leq 1.
\end{aligned}$$

Also the layer part $w(x,t)$ satisfies the bound as

$$||w(x,t)|| \leq C\exp\left(\frac{-\alpha(1-x)}{\varepsilon}\right).$$

To prove this bound we define the barrier function $\Psi^{\pm}(x,t)$ by

$$\Psi^{\pm}(x,t) = C\exp\left(\frac{-\alpha(1-x)}{\varepsilon}\right)\exp(t) \pm w(x,t).$$

At the boundaries and the initial stages we obtain

$$\begin{aligned}
\Psi^{\pm}(0,t) &= C\exp\left(\frac{-\alpha}{\varepsilon}\right)\exp(t) + w(0,t) \geq 0,\\
\Psi^{\pm}(1,t) &= C\exp(t) \pm w(1,t) \geq 0,\\
\Psi^{\pm}(x,0) &= C\exp\left(\frac{-\alpha(1-x)}{\varepsilon}\right) \pm w(x,0) \geq 0,
\end{aligned}$$

On the domain Q we have

$$\begin{aligned}\mathcal{L}\Psi^{\pm}(x,t) &= \Psi_t - \varepsilon\Psi_{xx} + a(x,t)\Psi_x + b(x,t)\Psi \\ &\geq C\exp\left(\frac{-\alpha(1-x)}{\varepsilon}\right)\exp(t)\left[1 - \frac{\alpha^2}{\varepsilon} + \frac{a(x,t)\alpha}{\varepsilon} + b(x,t)\right] \\ &\geq C\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\exp(t)[1+\beta] \geq 0.\end{aligned}$$

By the maximum principle, $\Psi^{\pm} \geq 0, \ \forall \ (x,t) \in \bar{Q}$. Therefore,

$$\begin{aligned}||w(x,t)|| &\leq C\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\exp(t) \\ &\leq C\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\exp(T) \\ &\leq C\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right), \ \forall \ (x,t) \ \in \bar{Q}.\end{aligned}$$

3.2 Spatial discretization with the FMFDM

In this section, we integrate the continuous problem (3.1.1)-(3.1.2) in space via FMFDM. Recall that problem (3.1.1)-(3.1.2) has a single boundary layer, so we employ the transition parameter σ

$$\sigma = \min\left\{\frac{1}{2}, \sigma_0\varepsilon\ln n\right\},$$

to divide the domain Ω into the sub-intervals $[0, 1-\sigma]$ and $[1-\sigma, 1]$, each with the mesh spacing $2(1-\sigma)/n$ and $2\sigma/n$ respectively. Thus if Δx_i is the mesh spacing, then it satisfies the piecewise function

$$\Delta x_i = x_i - x_{i-1} := \begin{cases} 2(1-\sigma)n^{-1}, & i = 1, 2, ..., n/2, \\ 2\sigma n^{-1}, & i = (n/2)+1, ..., n. \end{cases}$$

3.2.1 The FMFDM

Discretizing problem (3.1.1)-(3.1.2) in space on the mesh described above we have

$$\begin{aligned}\mathcal{L}^n U_i(t) \equiv \frac{dU_i(t)}{dt} - \frac{2\varepsilon}{\Delta x_i + \Delta x_{i+1}}\left(\frac{U_{i+1}(t) - U_i(t)}{\Delta x_{i+1}} - \frac{U_i(t) - U_{i-1}(t)}{\Delta x_i}\right) + a_i(t) \\ \times\left(\frac{U_i(t) - U_{i+1}(t)}{\Delta x_i}\right) + b_i(t)U_i(t) = f_i(t), \qquad i = 1, 2, ..., n-1. \quad (3.2.4)\end{aligned}$$

With the semi-discrete boundary and initial conditions

$$U_0(t) = \eta_0,\ \ U_n(t) = \eta_1 \ \text{ and } \ U_i(0) = \varphi_i, \tag{3.2.5}$$

respectively. In matrix notation, the scheme (3.2.4)-(3.2.5) takes the form

$$U'(t) + A(t)U(t) = F(t), \tag{3.2.6}$$

where $A(t) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ and $F(t)$ and $U(t)$ are in $\mathbb{R}^{n-1}$. We write the entries of $A(t)$ and $F(t)$ as

$$\begin{aligned}
A_{ii}(t) &= \frac{2\varepsilon}{\Delta x_i + \Delta x_{i+1}}\left(\frac{1}{\Delta x_{i+1}} + \frac{1}{\Delta x_i}\right) + \frac{a_i(t)}{\Delta x_i} + b_i(t), \quad && i = 1, 2, ..., n-1,\\
A_{i,i+1}(t) &= -\frac{2\varepsilon}{\Delta x_{i+1}(\Delta x_i + \Delta x_{i+1})}, && i = 1, 2, ..., n-2,\\
A_{i,i-1}(t) &= -\frac{2\varepsilon}{\Delta x_i(\Delta x_i + \Delta x_{i+1})} - \frac{a_i(t)}{\Delta x_i}, && i = 2, 3, ..., n-1,\\
F_1(t) &= f_1(t) + \left(\frac{2\varepsilon}{\Delta x_1(\Delta x_1 + \Delta x_2)} + \frac{a_1(t)}{\Delta x_1}\right)\eta_0, &&\\
F_i(t) &= f_i(t), && i = 2, 3, ..., n-2,\\
F_{n-1}(t) &= f_{n-1}(t) + \frac{2\varepsilon}{\Delta x_n(\Delta x_{n-1} + \Delta x_n)}\eta_1, &&
\end{aligned}$$

respectively. Next we analyse the scheme (3.2.4)-(3.2.5) for convergence.

3.2.2 Error analysis

Below we highlight some properties of the semi-discrete problem (3.2.4)-(3.2.5) which will be used in the error analysis. These lemmas have been adapted from [12, 57].

Lemma 3.2.1. *The operator $\mathcal{L}^n$ defined by the difference scheme (3.2.4)-(3.2.5) satisfies a semi-discrete maximum principle. That is if $\Phi_i(t)$ and $\Psi_i(t)$ are two mesh functions which satisfy $\Phi_0(t) < \Psi_0(t)$, $\Phi_n(t) < \Psi_n(t)$ and $\mathcal{L}^n\Phi_i(t) \le \mathcal{L}^n\Psi_i(t)$, $\forall\ (x_i, t) \in \Omega^n \times [0, T]$, then $\Phi_i(t) \le \Psi_i(t)$, $\forall\ (x_i, t) \in \bar{\Omega}^n \times [0, T]$.*

Proof. The coefficient matrix $A(t)$ in the linear system (3.2.6) has negative off diagonals and positive main diagonal entries. Thus it is an M-matrix with positive inverse. Therefore the solution $u_i(t),\ 1 \le i \le n-1, 0 \le t \le T$, exist and unique.

□

Lemma 3.2.2. *The solution $u_i(t)$ of the semi-discrete problem (3.2.4)-(3.2.5) is such that*

$$|u_i(t)| \le \beta^{-1} \max_{(x_i,t)\in\bar{\Omega}^n\times[0,T]} |\mathcal{L}^n u_i(t)| + \max_{(x,t)\in\bar{\Omega}^n\times[0,T]} \max(|\varphi_i|, \max(\eta_0,\eta_1)).$$

Proof. Let $p = \beta^{-1} \max_{(x_i,t)\in\bar{\Omega}^n\times[0,T]} |\mathcal{L}^n u_i(t)| + \max(|\varphi_0|, \max(\eta_0,\eta_1))$, and define the function $\Psi_i^{\pm}(t)$ by

$$\Psi_i^{\pm}(t) = p \pm u_i(t).$$

At the boundaries we have

$$\begin{aligned} \Psi_0^{\pm}(t) &= p \pm u_0(t) = p \pm \eta_0 \ge 0, \\ \Psi_n^{\pm}(t) &= p \pm u_n(t) = p \pm \eta_1 \ge 0. \end{aligned}$$

On the domain $0 < i < n$, we have

$$\begin{aligned} \mathcal{L}^n \Psi_i^{\pm}(t) &= (p \pm u_i(t))' - \varepsilon \left(\frac{p \pm u_{i+1}(t) - 2(p \pm u_i(t)) + p \pm u_{i-1}(t)}{\phi^2(\varepsilon,\Delta x,t)} \right) + a_i(t) \\ &\quad \times \left(\frac{p \pm u_i(t) - p \pm u_i(t)}{\Delta x} \right) + b_i(t)(p \pm u_i(t)) \\ &= (b_i(t))p \pm \mathcal{L}^n u_i(t) \\ &= (b_i(t))\beta^{-1} \max_{(x_i,t)\in\partial\Omega^n\times[0,T]} |\mathcal{L}^n u_i(t)| + \max(|\varphi_0|, \max(\eta_0,\eta_1)) \pm f_i(t) \\ &\ge 0, \ \ b_i(t) \ge \beta. \end{aligned}$$

From Lemma 3.2.1, $\Psi_i^{\pm}(t) \ge 0, \ \forall \ (x_i,t) \in \bar{\Omega}^n \times [0,T]$. This completes the proof. □

Following [21], we write the numerical solution as the sum

$$U_i(t) = V_i(t) + W_i(t).$$

Where $V_i(t)$ and $W_i(t)$ satisfy the respective problems

$$\mathcal{L}^n V_i(t) = f_i(t), \ \in \Omega^n \times (0,T], \tag{3.2.7}$$

$$V_i(0) = v_i(0), \ V_i(t) = v_i(t), \ i = 0, n, \ t \in (0,T],$$

$$\mathcal{L}^n W(t) = 0, \ \in \Omega^n \times (0,T], \tag{3.2.8}$$

$$W_i(0) = w_i(0), \ W_i(t) = w_i(t), \ i = 0, n, \ t \in (0,T].$$

The error is given as

$$\begin{aligned} |U_i(t) - u_i(t)| &= |(V_i(t) + W_i(t)) - (v_i(t) + w_i(t))| \\ &\leq |V_i(t) - v_i(t)| + |W_i(t) - w_i(t)|. \end{aligned}$$

Now we bound each term separately. We write the error of the regular component as

$$\begin{aligned} \mathcal{L}^n(V_i(t) - v_i(t)) &= f_i(t) - \mathcal{L}^n v_i(t) \\ &= (\mathcal{L} - \mathcal{L}^n) v_i(t) \\ &= -\varepsilon \left(\frac{\partial^2}{\partial x^2} - \delta^2 \right) v_i(t) + a_i(t) \left(\frac{\partial}{\partial x} - D^- \right) v_i(t) \end{aligned}$$

Using appropriate Taylor series expansions, we obtain the estimate

$$\begin{aligned} |\mathcal{L}^n(V_i(t) - v_i(t))| &\leq -\varepsilon \left(\frac{1}{3}(x_{i+1} - x_{i-1}) |(v_{xxx}(t))_i| \right) + a_i(t) \left(\frac{1}{2}(x_i - x_{i-1}) |(v_{xx}(t))_i| \right), \\ &\leq C(n^{-1})[\varepsilon |(v_{xxx}(t))_i| + |(v_{xx}(t))_i|] \\ &\leq C(\varepsilon n^{-1}(1 + \varepsilon^{-1})) \\ &\leq C n^{-1}(\varepsilon + 1) \\ &\leq C n^{-1}, \quad \text{since } \varepsilon << 1. \end{aligned}$$

Application of Lemma 3.2.2 results in

$$|(V_i(t) - v_i(t))| \leq C n^{-1}.$$

To bound the error in the singular component, it is important to note that its domain has the transition point σ, which is either $1/2$ or $\sigma_0 \varepsilon \ln n$. In the former case the mesh is uniform and the error can be estimated as the case of classical finite difference methods. Since our mesh is non-uniform we give the error bound when $\sigma = \sigma_0 \varepsilon \ln n$. In this case the spatial domain is divided into two sub-domains $[0, 1 - \sigma]$ and $[1 - \sigma, 1]$, each with the respective mesh spacing $2(1 - \sigma)/n$ and $2\sigma/n$. It is well known that the mesh is very coarse outside the layer region and so the derivatives of the solution in that region is very large. Therefore, we bound the error in each domain separately. Now we write the error in the non-layer region $[0, 1 - \sigma]$, as

$$|W_i(t) - w_i(t)| \leq |W_i(t)| + |w_i(t)|.$$

From the value of the transition point σ and $w(x,t) < C\exp(-\alpha(1-x)\varepsilon^{-1})$, for all $x \in [0, 1-\sigma]$ and $t \in (0,T]$ we have

$$\begin{aligned} |w_i(t)| &\leq C\exp(-\alpha(1-x)\varepsilon^{-1}) \\ &\leq C\exp(-\alpha\sigma\varepsilon^{-1}) \\ &\leq C\exp(-\alpha\frac{\varepsilon \ln n}{\alpha}\varepsilon^{-1}) \\ &\leq C\exp(-\ln n) \\ &\leq Cn^{-1}. \end{aligned}$$

To bound the numerical solution $W_i(t)$, we follow [58]. We define the barrier functions for all $t \in (0,T]$

$$Z_i(t) = \prod_{j=1}^{i}\left(1 + \frac{\alpha\Delta x_j}{\varepsilon}\right), \text{ on } \bar{\Omega}^n,$$

where $Z_0(t) = 1$, for $i = 0$. Note that $Z_i(t)$ is the first order Taylor series expansion of the boundary layer term $\exp(-\alpha(1-x)\varepsilon^{-1})$. Now for $1 \leq i \leq n-1$, the inequality

$$\mathcal{L}^n Z_i(t) \geq \frac{\alpha}{\varepsilon + \alpha\Delta x_i} Z_i(t), \tag{3.2.9}$$

holds and for $0 \leq i \leq n$, we have

$$\exp\left(-\alpha\frac{(1-x_i)}{\varepsilon}\right) \leq \prod_{j=i+1}^{n}\left(1 + \frac{\alpha\Delta x_j}{\varepsilon}\right)^{-1}. \tag{3.2.10}$$

Proof. The proof of (3.2.9) is as follows:

$$\begin{aligned} D^+Z_i(t) &= \frac{(1+\alpha\Delta x_{i+1}/\varepsilon)\left(\prod_{j=1}^{i}(1+\alpha\Delta x_j/\varepsilon)\right) - \left(\prod_{j=1}^{i}(1+\alpha\Delta x_j/\varepsilon)\right)}{\Delta x_{i+1}} \\ &= \frac{\left(\prod_{j=1}^{i}(1+\alpha\Delta x_j/\varepsilon)\right)[(1+\alpha\Delta x_{i+1}/\varepsilon) - 1]}{\Delta x_{i+1}} \\ &= \frac{\alpha}{\varepsilon}Z_i(t), \end{aligned} \tag{3.2.11}$$

$$D^-Z_i(t) = \frac{(1+\alpha\Delta x_i/\varepsilon)\left(\prod_{j=1}^{i-1}(1+\alpha\Delta x_j/\varepsilon)\right) - \left(\prod_{j=1}^{i-1}(1+\alpha\Delta x_j/\varepsilon)\right)}{\Delta x_i}$$

$$
\begin{aligned}
&= \frac{\left(\prod_{j=1}^{i-1}(1+\alpha\Delta x_j/\varepsilon)\right)[(1+\alpha\Delta x_i/\varepsilon)-1]}{\Delta x_i} \\
&= \frac{\alpha}{\varepsilon+\alpha\Delta x_i} Z_i(t), \qquad (3.2.12)
\end{aligned}
$$

From equations (3.2.11) and (3.2.12) it follows that

$$
\begin{aligned}
\mathcal{L}^n Z_i(t) &= \left[-\frac{2\varepsilon}{\Delta x_i+\Delta x_{i+1}}\left(\frac{\alpha}{\varepsilon}-\frac{\alpha}{\varepsilon+\alpha\Delta x_i}\right)+a_i(t)\left(\frac{\alpha}{\varepsilon+\alpha\Delta x_i}\right)+b_i(t)\right]Z_i(t) \\
&= \frac{1}{\varepsilon+\alpha\Delta x_i}\left[-\frac{2\alpha^2\Delta x_i}{\Delta x_i+\Delta x_{i+1}}+a_i(t)\alpha+b_i(t)(\varepsilon+\alpha\Delta x_i)\right]Z_i(t) \\
&= \frac{1}{\varepsilon+\alpha\Delta x_i}\left[-\frac{2\alpha^2\Delta x_i}{\Delta x_i+\Delta x_{i+1}}+\alpha^2+\beta(\varepsilon+\alpha\Delta x_i)\right]Z_i(t) \\
\mathcal{L}^n Z_i(t) &\geq \frac{C}{\varepsilon+\alpha\Delta x_i} Z_i(t), \qquad (3.2.13)
\end{aligned}
$$

giving us the right results. We give the proof of (3.2.10). For each j, we have

$$
\exp\left(-\frac{\alpha\Delta x_j}{\varepsilon}\right) = \exp\left(\frac{\alpha\Delta x_j}{\varepsilon}\right)^{-1} \leq \left(1+\frac{\alpha\Delta x_j}{\varepsilon}\right)^{-1}.
$$

Multiplying the inequalities for $j = i+1, ..., n$, completes the proof. □

Now we define the barrier function

$$
\begin{aligned}
\psi_i(t) &= C_1 \prod_{j=i+1}^{n}\left(1+\frac{\alpha\Delta x_j}{\varepsilon}\right)^{-1} \\
&= C_1 \prod_{j=1}^{n}\left(1+\frac{\alpha\Delta x_j}{\varepsilon}\right)^{-1} Z_i(t),
\end{aligned}
$$

and show that when C_1 is chosen to be sufficiently large $\psi_i(t)$ is a barrier function for $W_i(t)$. We observe that at the boundaries we obtain

$$
\begin{aligned}
W_0(t) &= |w(0,t)| \leq \exp(-\alpha/\varepsilon) = C\prod_{j=1}^{n}\exp(-\alpha\Delta x_j/\varepsilon) \leq C\prod_{j=1}^{n}\left(1+\frac{\alpha\Delta x_j}{\varepsilon}\right)^{-1} \\
W_n(t) &= |w(1,t)| \leq C,
\end{aligned}
$$

for $W_i(t)$ and

$$
\begin{aligned}
\psi_0(t) &= C_1\prod_{j=1}^{n}\left(1+\frac{\alpha\Delta x_j}{\varepsilon}\right)^{-1} \geq W_0(t) \\
\psi_n(t) &= C_1\prod_{j=1}^{n}\left(1+\frac{\alpha\Delta x_j}{\varepsilon}\right)^{-1} Z_n(t) \geq W_n(t),
\end{aligned}
$$

also holds when C_1 is chosen to be sufficiently large. At the interior mesh points

$$\mathcal{L}^n\psi_i(t) \geq \frac{C_1}{\varepsilon + \alpha\Delta x_i}\prod_{j=1}^{n}\left(1+\frac{\alpha\Delta x_j}{\varepsilon}\right)^{-1} Z_i(t) \geq \mathcal{L}^n W_i(t) = 0.$$

Since $W_i(t) \leq \psi_i(t)$ holds at the boundaries and $\mathcal{L}^n W_i(t) \leq \mathcal{L}^n\psi_i(t)$, for $i = 1, ..., n-1$ also holds, $\psi_i(t)$ is a discrete barrier function for $W_i(t)$, $\forall\ i$. Thus for $i = 0, ..., n/2$, we have

$$\begin{aligned}
\psi_i(t) \leq \psi_{n/2}(t) &= \prod_{j=1+n/2}^{n}\left(1+\frac{\alpha\Delta x_j}{\varepsilon}\right)^{-1}\\
&\leq C\exp(-\alpha(1-x_{n/2})/\varepsilon)\\
&= C\exp(-\alpha(1-(1-\sigma))/\varepsilon)\\
&= C\exp(-\alpha(\sigma))/\varepsilon)\\
&= C\exp(-\alpha(\varepsilon/\alpha\ln n)/\varepsilon)\\
&\leq Cn^{-1}.
\end{aligned}$$

Further on the sub-interval $(1-\sigma, 1]$, as indicated earlier the mesh is very fine and thus we estimate the error with consistency and barrier function argument. Now for all $n/2+1 \leq i \leq n-1$, we have

$$\begin{aligned}
|\mathcal{L}^n(W_i(t) - w_i(t))| &= -\varepsilon\left(\frac{\partial^2}{\partial x^2} - \delta^2\right) w_i(t) + a_i(t)\left(\frac{\partial}{\partial x} - D^-\right) w_i(t)\\
&\leq C(-\varepsilon\frac{1}{3}(x_{i+1} - x_{i-1})|(w_{xxx}(t))_i| + \frac{a_i(t)}{2}(x_i - x_{i-1})|(w_{xx}(t))_i|
\end{aligned}$$

Application of the bounds gives

$$\begin{aligned}
|\mathcal{L}^n(W_i(t) - w_i(t))| &\leq C\varepsilon^{-2}\Delta x_i\exp(-\alpha(1-x_i)/\varepsilon)\\
&\leq C\varepsilon^{-2}\sigma n^{-1},
\end{aligned}$$

with $|W_1(t) - w_1(t)| = 0$. Now we follow [21] to define the barrier functions

$$\Psi_i^{\pm}(t) = (x_i - (1-\sigma))C_1\varepsilon^{-2}\sigma n^{-1} + C_2 n^{-1} \pm W_i(t) - w_i(t).$$

It satisfies $\Psi^{\pm}_{\frac{n}{2}+1}(t) \geq 0$, $\Psi^{\pm}_n(t) \geq 0$, at the boundaries, and $\mathcal{L}^n\Psi_i^{\pm} \geq 0$, $\forall\ n/2+1 < i < n$. It follows from Lemma 3.2.1 that $\Psi_i^{\pm} \geq 0$, $\forall\ n/2+1 \leq i \leq n$, $t \in [0, T]$, therefore

$$|(W_i(t) - w_i(t))| \leq (x_i - (1-\sigma))C_1\varepsilon^{-2}\sigma n^{-1} + C_2 n^{-1}$$

$$\begin{aligned} &\leq C_1\varepsilon^{-2}\sigma^2 n^{-1} + C_2 n^{-1} \\ &\leq Cn^{-1}(\ln n)^2. \end{aligned}$$

Combing the results in each sub-domain gives

$$|W_i(t) - w_i(t)| \leq Cn^{-1}\ln n^2,$$

as the error in the singular component. Addition of the error of the regular component gives the total error bound

$$\begin{aligned} |U_i(t) - u_i(t)| &\leq |(V-v)(x_i,t)| + |(W-w)(x_i,t)| \\ &\leq Cn^{-1} + Cn^{-1}(\ln n)^2 \\ &\leq Cn^{-1}\ln n^2. \end{aligned}$$

Lemma 3.2.3. *The error of the spatial discretization with the fitted mesh finite difference method satisfies*

$$\max_{0<\varepsilon<1}\max_{0\leq i\leq n}|U_i(t) - u_i(t)| \leq Cn^{-1}(\ln n)^2.$$

3.3 Spatial discretization with the FOFDM

This section is concerned with the spatial discretization of problem (3.1.1)-(3.1.2) using the FOFDM.

3.3.1 The FOFDM

Let n to be a positive integer and consider the following partition of the interval $[0,1]$

$$x_0 = 0,\ x_i = x_0 + i\Delta x,\ \ i = 1(1)n-1,\ \ \Delta x = x_i - x_{i-1},\ x_n = 1.$$

We use the notation in Chapter 2 and perform the space discretization as follows

$$\begin{aligned} \mathcal{L}^{\Delta x}U_i(t) \equiv \frac{dU_i(t)}{dt} - \varepsilon\frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{\phi_i^2(\varepsilon,\Delta x,t)} + a_i(t)\frac{U_i(t) - U_{i-1}(t)}{\Delta x} \\ +b_i(t)U_i(t) = f_i(t), \quad i = 1,2,...,n-1, \end{aligned} \tag{3.3.14}$$

subject to the semi-discrete boundary and initial conditions

$$U_0(t) = \eta_0,\ \text{ and }\ U_n(t) = \eta_1,\ \text{ and }\ U_i(0) = \varphi_i. \tag{3.3.15}$$

The denominator function $\phi_i^2(\varepsilon, \Delta x, t)$ is given as

$$\phi_i^2(\varepsilon, \Delta x, t) = \frac{\varepsilon \Delta x}{a_i(t)} \exp\left(\frac{a_i(t)\Delta x}{\varepsilon} - 1\right).$$

The above systems of initial value problems in equation (3.3.14) can be represented in the matrix notation

$$U'(t) = A(t)U(t) + F(t). \tag{3.3.16}$$

Here $A(t)$ is tridiagonal matrix $\in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ and $U(t)$ and $F(t) \in \mathbb{R}^{(n-1)}$. The entries of $A(t)$ and $F(t)$ are given as:

$$\begin{aligned}
A_{ii}(t) &= \frac{2\varepsilon}{\phi_i^2(\varepsilon, \Delta x, t)} + \frac{a_i(t)}{\Delta x} + b_i(t), && i = 1, 2, ..., n-1, \\
A_{i,i+1}(t) &= -\frac{\varepsilon}{\phi_i^2(\varepsilon, \Delta x, t)}, && i = 1, 2, ..., n-2, \\
A_{i,i-1}(t) &= -\frac{\varepsilon}{\phi_i^2(\varepsilon, \Delta x, t)} - \frac{a_i(t)}{\Delta x}, && i = 2, ..., n-1, \\
F_1(t) &= f_1(t) + \left(\frac{\varepsilon}{\phi_1^2(\varepsilon, \Delta x, t)} + \frac{a_1(t)}{\Delta x}\right)\eta_0, \\
F_i(t) &= f_i(t) && i = 2, 3, ..., n-2, \\
F_{n-1}(t) &= f_{n-1}(t) + \frac{\varepsilon}{\phi_{n-1}^2(\varepsilon, \Delta x, t)}\eta_1,
\end{aligned}$$

respectively. Now we analyse this spatial discretization for convergence. Note that the semi-discrete operator $\mathcal{L}^{\Delta x}$ also satisfies the maximum principle and the uniform stability estimate. These properties and their proofs are similar to that of the Lemmas 3.2.1 and 3.2.2.

3.3.2 Error analysis

The truncation error of the scheme (3.3.14)-(3.3.15) is

$$\begin{aligned}
\mathcal{L}^{\Delta x}(U_i(t) - u_i(t)) &= f_i(t) - \mathcal{L}^{\Delta x} u_i(t) \\
&= (\mathcal{L} - \mathcal{L}^{\Delta x})u_i(t) \\
&\quad -\varepsilon(u_{xx}(t))_i + a_i(t)(u_x(t))_i + \varepsilon\frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{\phi_i^2(\varepsilon, \Delta x, t)} \\
&\quad -a_i(t)\frac{u_i(t) - u_{i-1}(t)}{\Delta x}.
\end{aligned}$$

Using Taylor series expansion of $u_{i+1}(t)$ and $u_{i-1}(t)$ leads to

$$\begin{aligned}\mathcal{L}^{\Delta x}(U_i(t)-u_i(t)) &= -\varepsilon(u_{xx}(t))_i + \frac{\varepsilon}{\phi_i^2(\varepsilon,\Delta x,t)}\left(\Delta x^2(u_{xx}(t))_i + \frac{\Delta x^4}{12}(u_{xxxx}(t))_i\xi_i\right), \\ &\quad + \frac{a_i(t)\Delta x}{2}(u_{xx}(t))_i, \quad \xi_i \in (u_{i+1}(t), u_{i-1}(t)).\end{aligned}$$

Again using a truncated Taylor series expansions of $\phi_i^{-2} = 1/\Delta x^2 - a_i(t)/(2\Delta x\varepsilon) + a_i^2(t)/(12\varepsilon^2)$, we obtain

$$\begin{aligned}\mathcal{L}^{\Delta x}(U_i(t)-u_i(t)) &= -\varepsilon(u_{xx}(t))_i + \left(\Delta x^2(u_{xx}(t))_i + \frac{\Delta x^4}{12}(u_{xxxx}(t))_i\xi_i\right) \\ &\quad \times\left(\frac{\varepsilon}{\Delta x^2} - \frac{a_i(t)}{2\Delta x} + \frac{a_i^2(t)}{12\varepsilon}\right) + \frac{a_i(t)\Delta x}{2}(u_{xx}(t))_i \\ &= -\varepsilon(u_{xx}(t))_i + \left(\frac{\varepsilon}{\Delta x^2} - \frac{1}{2}\left(\frac{a_i(t)-a_{i-1}(t)}{\Delta x}\right) + \frac{a_i^2(t)}{12\varepsilon}\right) \\ &\quad \times\left(\Delta x^2(u_{xx}(t))_i + \frac{\Delta x^4}{12}(u_{xxxx}(t))_i\xi_i\right) + \frac{a_i(t)\Delta x}{2}(u_{xx}(t))_i \\ &= \varepsilon\frac{\Delta x^2}{12}(u_{xxxx}(t))_i\xi_i - \frac{(a_x(t))_i\Delta x^2(u_{xx}(t))_i}{2} - \frac{(a_x(t))_i\Delta x^4(u_{xxxx}(t))_i}{24} \\ &\quad + \frac{a_i^2(t)\Delta x^2(u_{xx}(t))_i}{12\varepsilon} + \frac{a_i^2(t)\Delta x^4(u_{xxxx}(t))_i}{144\varepsilon}\xi_i + \frac{a_i(t)\Delta x(u_{xx}(t))_i}{2} \\ &= \frac{a_i(t)\Delta x(u_{xx}(t))_i}{2} + \left(-\frac{(a_x(t))_i(u_{xxxx}(t))_i}{24} + \frac{a_i^2(t)(u_{xxxx}(t))_i}{144\varepsilon}\xi_i\right)\Delta x^4 \\ &\quad + \left(\frac{\varepsilon}{12}(u_{xxxx}(t))_i\xi_i - \frac{(a_x(t))_i(u_{xx}(t))_i}{2} + \frac{a_i^2(t)(u_{xx}(t))_i}{12\varepsilon}\right)\Delta x^2\end{aligned}$$

Applying the bounds of the solution and its derivatives (3.1.3) and Lemma 2.3.1 gives

$$|\mathcal{L}^{\Delta x}(U_i(t)-u_i(t))| \le \frac{a_i(t)\Delta x}{2} + \left(\frac{\varepsilon}{12} - \frac{(a_x(t))_i}{2} + \frac{a_i^2(t)}{12\varepsilon}\right)\Delta x^2 + \left(-\frac{(a_x(t))_i}{24} + \frac{a_i^2(t)u}{144\varepsilon}\xi_i\right)\Delta x^4.$$

From the relation $\Delta x > \Delta x^2 > \Delta x^4$ and the uniform stability estimate 3.2.2, we obtain

$$U_i(t) - u_i(t)| \le C\Delta x.$$

Lemma 3.3.1. *Let $u_i(t)$ be the exact solution of (3.1.1)-(3.1.2) and $U_i(t)$ the solution of (3.3.14)-(3.3.15) at $x = x_i$. Then we have*

$$\max_{0<\varepsilon\le 1}\max_{0\le i\le n}|U_i(t)-u_i(t)| \le C\Delta x,$$

where C is a constant independent of Δx and ε.

3.4 Time discretization

In this section, we integrate the systems of IVPs which resulted from the spatial discretization. Similar to the early chapter we use the backward Euler integration method on a uniform mesh. Now we write the fully discrete scheme as

$$\frac{U^k - U^{k-1}}{\tau} + A(t_k)U^k = F(t_k), \quad \text{for } k = 1, ..., K, \tag{3.4.17}$$

with initial condition $U(0) = \varphi$. Further simplification of equation (4.4.29) gives

$$U^k = (\mathbf{I} + \tau A(t_k))^{-1} (\tau F(t_k) + U^{k-1}).$$

Since the backward Euler has already been analysed for convergence in Chapter 2 we combine the result with the Lemmas 3.2.3 and 3.3.1 to give the main results in this chapter.

Theorem 3.4.1. *Let $u \in \mathcal{C}^{4,2}(\bar{Q})$ be the exact solution of the continuous problem (3.1.1)-(3.1.2) and U_i^k be the numerical solution obtained via the FMDML (3.2.4)-(3.2.5) along with (3.4.17) or the FOFDML (3.3.14)-(3.3.15) along with (3.4.17). Then the errors of these methods are as follows:*

$$\sup_{0<\varepsilon\leq 1} \max_{0\leq i\leq n; 0\leq k\leq K} ||U_i^k - u_i^k|| \leq C(n^{-1}(\ln n)^2 + \tau), \quad \textit{for the FMFDML}$$

and

$$\sup_{0<\varepsilon\leq 1} \max_{0\leq i\leq n; 0\leq k\leq K} ||U_i^k - u_i^k|| \leq C(\Delta x + \tau), \quad \textit{for the FOFDML.}$$

Next we perform numerical simulations to support these theoretical findings.

3.5 Numerical example

Here we validate Theorem 3.4.1 with an example. We compute the maximum pointwise error and the rate of convergence for different values of n, K and ε. These results are then displayed in table formats.

Example 3.5.1. *[15] Consider the problem*

$$u_t - \varepsilon u_{xx} + \left(1 + x^2 + \frac{1}{2}\sin(\pi x)\right) u_x + (1 + x^2 + \sin(\pi t/2))u = f, \tag{3.5.18}$$

$$f = x^3(1-x)^3 + t(1-t)\sin(\pi t), \quad (x,t) \in \Omega \times (0,1],$$

$$u(x,0) = u(0,t) = u(1,t) = 0. \tag{3.5.19}$$

The exact solution of this problem is unknown. Thus to calculate the error we use the formula

$$E^{\varepsilon}_{n,\tau} = \max_{0\le i\le n; 0\le k\le K} |U^{k;K}_{i;n} - U^{k;2K}_{i;2n}|, \tag{3.5.20}$$

where $U^{k;K}_{i;n}$ is the numerical solution and $U^{k;2K}_{i;2n}$ is also a numerical solution but on the mesh $\mu(2n, 2K)$. Note the we have used the $2n$ and the $2K$ because the FMFDL and the FOFDML are first order accurate in space (except for a logarithmic factor in the case of FMFDML) and first order accurate in the time variable. Also, we compute the rate of convergence using the formula

$$r_l = \log_2\left(E^{\varepsilon}_{n,\tau}/E^{\varepsilon}_{2n,\tau/2}\right), \quad l = 1, 2, ... \tag{3.5.21}$$

Table 3.1: Maximum pointwise error for Example 3.5.1 using the FMFDML

ε	$n = K = 32$	64	128	256	512	1024
10^{-1}	$1.81E-02$	$9.03E-03$	$4.35E-03$	$2.14E-03$	$1.10E-03$	$5.84E-04$
10^{-2}	$2.19E-02$	$1.67E-02$	$1.18E-02$	$7.25E-03$	$3.81E-03$	$1.79E-03$
10^{-3}	$1.78E-02$	$1.20E-02$	$8.23E-03$	$6.01E-03$	$4.72E-03$	$3.63E-03$
10^{-4}	$1.72E-02$	$1.10E-02$	$6.66E-03$	$3.94E-03$	$2.42E-03$	$1.68E-03$
10^{-5}	$1.71E-02$	$1.09E-02$	$6.48E-03$	$3.67E-03$	$2.03E-03$	$1.13E-03$
10^{-6}	$1.71E-02$	$1.09E-02$	$6.46E-03$	$3.64E-03$	$2.00E-03$	$1.07E-03$
10^{-7}	$1.71E-02$	$1.09E-02$	$6.46E-03$	$3.64E-03$	$2.00E-03$	$1.07E-03$
10^{-8}	$1.71E-02$	$1.09E-02$	$6.46E-03$	$3.64E-03$	$2.00E-03$	$1.07E-03$
10^{-9}	$1.71E-02$	$1.09E-02$	$6.46E-03$	$3.64E-03$	$2.00E-03$	$1.07E-03$
10^{-10}	$1.71E-02$	$1.09E-02$	$6.46E-03$	$3.64E-03$	$2.00E-03$	$1.07E-03$

Table 3.2: Maximum pointwise error for Example 3.5.1 using the FOFDML

ε	$n = K = 32$	64	128	256	512	1024
10^{0}	$2.45E-04$	$1.25E-04$	$6.30E-05$	$3.16E-05$	$1.59E-05$	$7.94E-06$
10^{-1}	$1.23E-03$	$6.20E-04$	$3.11E-04$	$1.56E-04$	$7.79E-05$	$3.89E-05$
10^{-2}	$2.29E-03$	$1.09E-03$	$4.99E-04$	$2.34E-04$	$1.13E-04$	$5.54E-05$
10^{-3}	$2.51E-03$	$1.39E-03$	$7.35E-04$	$3.61E-04$	$1.62E-04$	$7.07E-05$
10^{-4}	$2.51E-03$	$1.39E-03$	$7.39E-04$	$3.81E-04$	$1.93E-04$	$9.72E-05$
10^{-5}	$2.51E-03$	$1.39E-03$	$7.39E-04$	$3.81E-04$	$1.93E-04$	$9.73E-05$
10^{-6}	$2.51E-03$	$1.39E-03$	$7.39E-04$	$3.81E-04$	$1.93E-04$	$9.73E-05$
10^{-7}	$2.51E-03$	$1.39E-03$	$7.39E-04$	$3.81E-04$	$1.93E-04$	$9.73E-05$
10^{-8}	$2.51E-03$	$1.39E-03$	$7.39E-04$	$3.81E-04$	$1.93E-04$	$9.73E-05$
10^{-9}	$2.51E-03$	$1.39E-03$	$7.39E-04$	$3.81E-04$	$1.93E-04$	$9.73E-05$
10^{-10}	$2.51E-03$	$1.39E-03$	$7.39E-04$	$3.81E-04$	$1.93E-04$	$9.73E-05$

Table 3.3: Rate of convergence for Example 3.5.1 using the FMFDML

ε	r_1	r_2	r_3	r_4	r_5
10^{0}	0.95	0.98	0.99	0.99	1.00
10^{-1}	0.79	0.83	0.93	0.97	0.99
10^{-2}	0.78	0.83	0.86	0.89	0.90
10^{-3}	0.77	0.82	0.85	0.88	0.90
$10-4$	0.77	0.81	0.85	0.88	0.90
10^{-5}	0.77	0.81	0.85	0.88	0.90
10^{-6}	0.77	0.81	0.85	0.88	0.90
10^{-7}	0.77	0.81	0.85	0.88	0.90
10^{-8}	0.77	0.81	0.85	0.88	0.90
10^{-9}	0.77	0.81	0.85	0.88	0.90
10^{-10}	0.77	0.81	0.85	0.88	0.90

Table 3.4: Rate of convergence for Example 3.5.1 using the FOFDML

ε	r_1	r_2	r_3	r_4	r_5
10^{0}	0.98	0.99	0.99	1.00	1.00
10^{-1}	0.99	1.00	1.00	1.00	1.00
10^{-2}	1.07	1.13	1.09	1.05	1.03
10^{-3}	0.85	0.92	1.03	1.16	1.19
10^{-4}	0.85	0.92	0.96	0.98	0.99
10^{-5}	0.85	0.92	0.96	0.98	0.99
10^{-6}	0.85	0.92	0.96	0.98	0.99
10^{-7}	0.85	0.92	0.96	0.98	0.99
10^{-8}	0.85	0.92	0.96	0.98	0.99
10^{-9}	0.85	0.92	0.96	0.98	0.99
10^{-10}	0.85	0.92	0.96	0.98	0.99

3.6 Conclusion

This chapter was devoted to one-dimensional convection-diffusion problems. First, we followed the idea of [12] in designing a FMFDML to integrate these problems. The FMDML consists of a FMFDM for the spatial variable and the backward Euler method to integrate the time variable. We showed that this method is of order $\mathcal{O}(Cn^{-1}(\ln n)^2 + \tau)$, where n is the number of sub-intervals in space and τ is the temporal step size. Then, we developed a FOFDML which is also a combination of a FOFDM and the backward Euler method for the space and the time respectively. A rigorous error analysis showed that the present FOFDML is uniformly convergent with respect to the perturbation parameter ε. In other words, the method satisfies $\mathcal{O}(\Delta x + \tau)$, where Δx is the discretisation parameter in space. For illustrative purposes, simulations on a test example were conducted and numerical results thereof confirmed the theoretical findings.

Chapter 4

Methods of Lines for Two-Dimensional Reaction-Diffusion Problems

In the early chapters, we introduced SPPs, numerical methods for their solutions in the framework of the method of lines. Also, we used the FMFDML and the FOFDML to integrate both one-dimensional time-dependent reaction-diffusion and convection-diffusion problems.

In this chapter and in the next, we extend the methods of chapters 2 and 3 to solve two-dimensional time-dependent reaction-diffusion problems. We discretize in space with the FMFDM and the FOFDM to obtain semi-discrete ordinary differential equations (ODE) in time and then solve the ODEs with the backward Euler method. We give the theoretical estimate of the errors in connection with the spatial discretization and also compute numerical examples for illustration purpose.

4.1 Continuous problem

We consider the time-dependent reaction-diffusion problem

$$\mathcal{L}u \equiv \frac{\partial u}{\partial t} - \varepsilon \Delta u + b(x,y,t)u = f(x,y,t), \quad (x,y,t) \in Q \equiv \Omega = (0,1)^2 \times (0,T], \quad (4.1.1)$$

with the initial and boundary conditions

$$u(x,y,0)=\varphi(x,y),\ \ (x,y)\in\bar{\Omega},\ \ u(x,y,t)=g(x,y,t)\in\ \partial\Omega\times(0,T]. \tag{4.1.2}$$

Here $\varepsilon\in(0,1]$ and $b(x,y,t)$, $f(x,y,t)$ are sufficiently smooth and the coefficient function, $b(x,y,t)$ satisfies $b(x,y,t)\geq\beta>0,\ \forall\ (x,y,t)\in Q$. Also, we impose the compatibility conditions

$$\begin{aligned}
g(x,y,0) &= \varphi(x,y),\ \text{in}\ \partial\Omega,\\
\frac{\partial g}{\partial t}(x,y,0) &= \varepsilon\bigtriangleup\varphi-b(x,y,0)\varphi+f(x,y,0),\ \text{in}\ \partial\Omega,\\
\frac{\partial^2 g}{\partial t^2}(x,y,0) &= (-\varepsilon\bigtriangleup+b(x,y,0))^2\varphi+\frac{\partial}{\partial t}f(x,y,0)+(\varepsilon\bigtriangleup-b(x,y,0))\,f(x,y,0),\ \text{in}\ \partial\Omega,\\
f(x,y,0) &= \left(\frac{\partial}{\partial t}-\varepsilon\bigtriangleup+b(x,y,0)\right)g(x,y,t),\ \text{in}\ (0,1)\times(0,1)\times(0,T],
\end{aligned}$$

for the exact solution to be differentiable at the corners of Q. Even for the smooth data and the compatibility conditions, solutions to problem (4.1.1)-(4.1.2) experience some abrupt changes (layers) around the boundaries of Ω. Figure 4.1 illustrates the layers on the domain Ω.

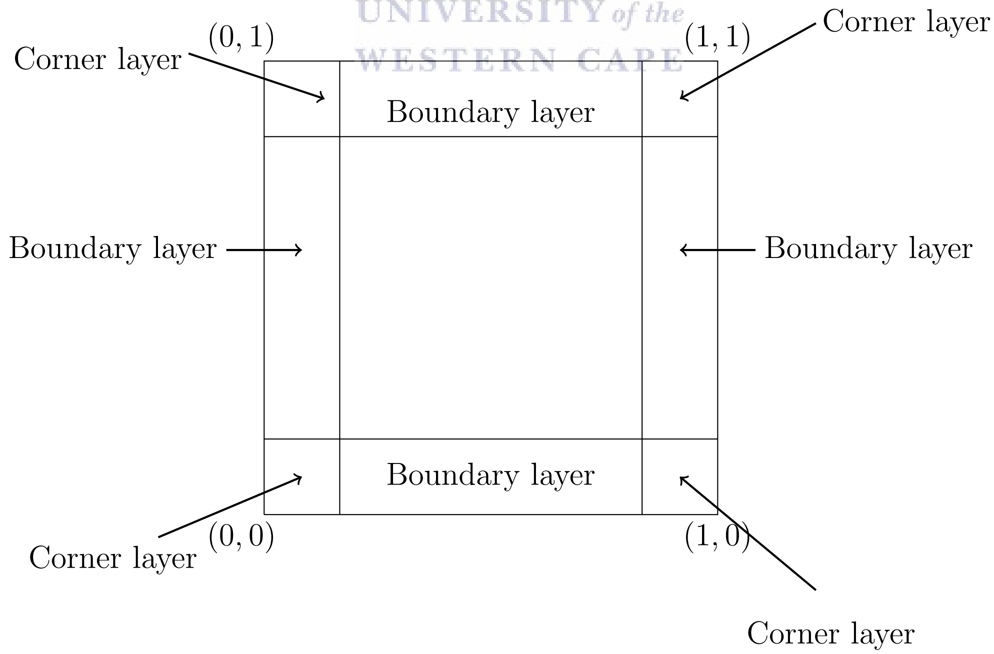

Figure 4.1: Parabolic layers for the elliptic reaction-diffusion problem.

Under the smoothness assumption and compatibility conditions the differential operator $\mathcal{L} = \partial/\partial t - \varepsilon\left(\partial^2/\partial x^2 + \partial/\partial y^2\right) + b\mathbf{I}$, satisfies the continuous maximum principle as well as the uniform stability estimate below. These properties have been adapted from [34] and they show the existence and uniqueness of the solution to problem (4.1.1)-(4.1.2).

Lemma 4.1.1. *Continuous maximum principle. Let* $\psi \in \mathcal{C}^{4,2}(\bar{Q})$ *be such that* $\psi \geq 0$*, on* ∂Q*. Then* $\mathcal{L}(x,y,t) \geq 0,\ \forall\ (x,y,t)\ \in Q$ *implies that* $\psi(x,y,t) \geq 0,\ \forall\ (x,y,t) \in \bar{Q}$.

Proof. Suppose $(x^*,y^*,t^*) \in \bar{Q}$ and satisfies $\psi(x^*,y^*,t^*) = \min\limits_{(x,y,t)\in\bar{Q}} \psi(x^*,y^*,t^*)$ and $\psi(x^*,y^*,t^*) \leq 0$. It is evident that $(x^*,y^*,t^*) \in Q$. We know $\psi_t(x^*,y^*,t^*) = 0,\ \ \psi_{xx}(x^*,y^*,t^*) \geq 0,\ \ \psi_{yy}(x^*,y^*,t^*) \geq 0$, thus we observe that

$$\mathcal{L}(x^*,y^*,t^*) = \psi_t(x^*,y^*,t^*) - \varepsilon\Delta\psi(x^*,y^*,t^*) + b(x^*,y^*,t^*)\psi(x^*,y^*,t^*) < 0,$$

which is a contradiction, therefore, $\psi(x,y,t) \geq 0,\ \forall\ (x,y,t) \in \bar{Q}$. □

Lemma 4.1.2. *If* u *is the solution of problem (4.1.1)-(4.1.2) then it satisfies the bound*

$$u \leq \beta^{-1}||f|| + \max(\varphi(x,y), g(x,y,t)).$$

Proof. We define the barrier functions $\Psi^{\pm}$ by

$$\Psi^{\pm}(x,t) = \beta^{-1}||f|| + \max(\varphi(x,y), g(x,y,t)) \pm u(x,y,t).$$

The values of $\Psi^{\pm}(x,t)$ at the initial stage and the boundaries are

$$\begin{aligned}
\Psi^{\pm}(x,y,0) &= \beta^{-1}||f|| + \max(\varphi(x,y), g(x,y,t)) \pm u(x,y,0) \\
&= \beta^{-1}||f|| + \max(\varphi(x,y), g(x,y,t)) \pm \varphi(x,y) \\
&\geq 0, \\
\Psi^{\pm}(x,y,t) &= \beta^{-1}||f|| + \max(\varphi(x,y), g(x,y,t)) \pm u(x,y,t) \\
&= \beta^{-1}||f|| + \max(\varphi(x,y), g(x,y,t)) \pm g(x,y,t) \\
&\geq 0.
\end{aligned}$$

Further on the domain Q, we have

$$\begin{aligned}
\mathcal{L}\Psi^{\pm}(x,y,t) &= \Psi^{\pm}_t(x,y,t) - \varepsilon\Delta\Psi^{\pm}(x,y,t) + b(x,y,t)\Psi^{\pm}(x,y,t) \\
&= b(x,y,t)(\beta^{-1}||f|| + \max(\varphi(x,y), g(x,y,t)) \pm \mathcal{L}u(x,y,t) \\
&= b(x,y,t)(\beta^{-1}||f|| + \max(\varphi(x,y), g(x,y,t)) \pm f(x,y,t) \\
&\geq 0, \quad \text{since} \quad b(x,y,t) \geq \beta.
\end{aligned}$$

Thus from Lemma 4.1.1, $\Psi^{\pm}(x,y,t) \geq 0, \ \forall\ (x,y,t) \in \bar{Q}$. □

In reference to lemma 4.1.1 and 4.1.2 the exact solution $u(x,y,t)$ of problem (4.1.1)-(4.1.2) and its derivatives satisfy [40]

$$\begin{aligned}
|u_{xx}(x,y,t)| &\leq (1 + \varepsilon^{-1}(\exp\left(-\beta x/\sqrt{\varepsilon}\right)) + \exp\left(-\beta(1-x)/\sqrt{\varepsilon}\right)\right), \\
|u_{yy}(x,y,t)| &\leq (1 + \varepsilon^{-1}(\exp\left(-\beta y/\sqrt{\varepsilon}\right)) + \exp\left(-\beta(1-y)/\sqrt{\varepsilon}\right)\right).
\end{aligned}$$

The exact solution can be written as [12]

$$u = v + \sum_{p=1}^{4} w_p + \sum_{p=1}^{4} z_p, \tag{4.1.3}$$

where v is the regular component, w_p, $p = 1,..,4$ are the edge layer functions around the sides $x = 0$, $y = 0$, $x = 1$ and $y = 1$ and $z_p = 1,..,4$ are the corner layer functions around the four corners $(0,0)$ $(1,0)$, $(0,1)$, $(1,1)$ respectively. Furthermore, they satisfy the respective bounds

$$\frac{\partial^{i+j+k} v(x,y,t)}{\partial x^i \partial y^j \partial t^k} \leq C \tag{4.1.4}$$

$$\frac{\partial^{i+j+k} w_1(x,y,t)}{\partial x^i \partial y^j \partial t^k} \leq C\varepsilon^{-i/2} \exp\left(-\sqrt{\beta/\varepsilon} x\right) \tag{4.1.5}$$

$$\frac{\partial^{i+j+k} w_2(x,y,t)}{\partial x^i \partial y^j \partial t^k} \leq C\varepsilon^{-i/2} \exp\left(-\sqrt{\beta/\varepsilon}(1-x)\right) \tag{4.1.6}$$

$$\frac{\partial^{i+j+k} w_3(x,y,t)}{\partial x^i \partial y^j \partial t^k} \leq C\varepsilon^{-j/2} \exp\left(-\sqrt{\beta/\varepsilon} y\right) \tag{4.1.7}$$

$$\frac{\partial^{i+j+k} w_4(x,y,t)}{\partial x^i \partial y^j \partial t^k} \leq C\varepsilon^{-j/2} \exp\left(-\sqrt{\beta/\varepsilon}(1-y)\right) \tag{4.1.8}$$

$$\frac{\partial^{i+j+k} z_1(x,y,t)}{\partial x^i \partial y^j \partial t^k} \leq C\varepsilon^{-(i+j)/2} \min\left\{\exp\left(-\sqrt{\beta/\varepsilon x}\right), \exp\left(-\sqrt{\beta/\varepsilon y}\right)\right\} \tag{4.1.9}$$

$$\frac{\partial^{i+j+k} z_2(x,y,t)}{\partial x^i \partial y^j \partial t^k} \leq C\varepsilon^{-(i+j)/2} \min\left\{\exp\left(-\sqrt{\beta/\varepsilon}(1-x)\right), \exp\left(-\sqrt{\beta/\varepsilon y}\right)\right\} \tag{4.1.10}$$

$$\frac{\partial^{i+j+k} z_3(x,y,t)}{\partial x^i \partial y^j \partial t^k} \leq C\varepsilon^{-(i+j)/2} \min\left\{\exp\left(-\sqrt{\beta/\varepsilon x}\right), \exp\left(-\sqrt{\beta/\varepsilon}(1-y)\right)\right\} \tag{4.1.11}$$

$$\frac{\partial^{i+j+k} z_4(x,y,t)}{\partial x^i \partial y^j \partial t^k} \leq C\varepsilon^{-(i+j)/2} \min\left\{\exp\left(-\sqrt{\beta/\varepsilon}(1-x)\right), \exp\left(-\sqrt{\beta/\varepsilon}(1-y)\right)\right\}, \tag{4.1.12}$$

where $i + j + 2k \leq 4$. In the next section, we discretize the space variables with the FMFDM.

4.2 Spatial discretization with the FMFDM

In this section, we discretize in space the continuous problem (4.1.1)-(4.1.2). Note that the FMFDM which is considered in this chapter is the one derived by [12]. Problems of types (4.1.1)-(4.1.2) are known to have two boundary layers each at the x and the y directions. Thus to obtain the mesh for the entire spatial domain $\Omega = (0,1)^2$, we use the tensor product of the two one-dimensional meshes in x and y directions. That is $\Omega^n = \mathbb{I}_{x,n} \otimes \mathbb{I}_{y,n}$, where $\mathbb{I}_{x,n} = \{0 = x_0 < ... < x_n = 1\}$, $\mathbb{I}_{y,m} = \{0 = y_0 < ... < y_m = 1\}$. Supposing $n = m \geq 4$ we use the transition parameter $\sigma_x = \sigma_y$, given by

$$\sigma_x = \sigma_y = \min\left\{\frac{1}{4}, 2\sqrt{\frac{\varepsilon}{\beta}} \ln n\right\},$$

to sub-divide the x domain $(0,1)$ into $[0, \sigma_x]$, $[\sigma_x, 1-\sigma_x]$ and $[1-\sigma_x, 1]$. This distributes the mesh into $n/2 + 1$ uniform mesh points in the sub-interval $[\sigma_x, 1-\sigma_x]$, and $n/4 + 1$ uniform mesh points in the sub-intervals $[0, \sigma_x]$, $[1-\sigma_x, 1]$. The mesh spacing Δx_i in this case is given by

$$\Delta x_i = x_i - x_{i-1} := \begin{cases} 4\sigma n^{-1}, & i = 1, 2, ..., n/4, \\ 2(1-2\sigma)n^{-1}, & i = n/4+1, ..., 3n/4, \\ 4\sigma n^{-1} & i = 3n/4+1, ..., n. \end{cases}$$

The mesh in y direction can be obtain in an analogous manner and since we have already assumed that $n = m$, $\sigma_x = \sigma_y$, it implies that $\Delta y_j = \Delta x_i$. Figure 4.2 gives a picture of the mesh on the square domain Ω.

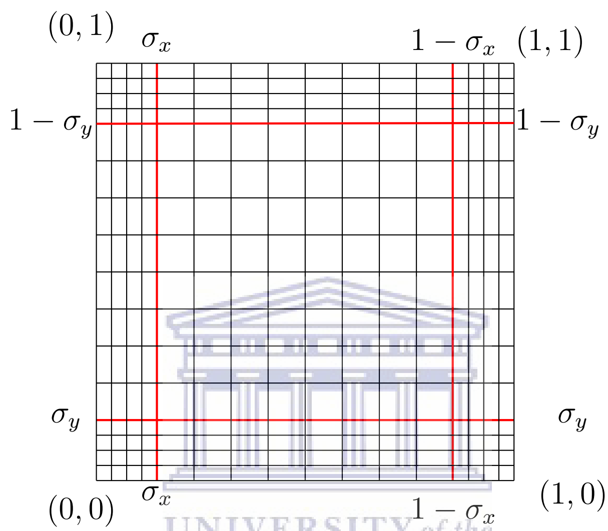

Figure 4.2: Shishkin mesh for $n = 16$ for an elliptic reaction-diffusion problem

4.2.1 The FMFDM

Now we adopt the notation $\omega_{ij}(t) = \omega(x_i, y_j, t)$ and denote by $U_{ij}(t)$ the approximation of $u(x_i, y_j, t)$. We perform the discretization as follows

$$\begin{aligned} \mathcal{L}^{n,m} U_{ij}(t) \quad \equiv \quad & \frac{dU_{ij}(t)}{dt} - \frac{2\varepsilon}{\Delta x_i + \Delta x_{i+1}} \left(\frac{U_{i+1,j}(t) - U_{ij}(t)}{\Delta x_{i+1}} - \frac{U_{ij}(t) - U_{i-1,j}(t)}{\Delta x_i} \right) \\ & - \frac{2\varepsilon}{\Delta y_j + \Delta y_{j+1}} \left(\frac{U_{i,j+1}(t) - U_{ij}(t)}{\Delta y_{j+1}} - \frac{U_{ij}(t) - U_{i,j-1}(t)}{\Delta y_j} \right) + b_{ij}(t) U_{ij}(t) \\ & = f_{ij}(t), \quad i = 1(1)n-1, \;\; j = 1(1)m-1. \end{aligned} \tag{4.2.13}$$

With the semi-discrete boundary and initial conditions

$$U_{0j}(t) \;\; = \;\; g_{0j}(t), \; U_{nj}(t) = g_{nj}(t), \; U_{i0}(t) = g_{i0}(t), \; U_{im}(t) = g_{im}(t), \;\; (x_i, y_j) \in \partial\Omega^n,$$

$$t \in [0,T], \quad U_{ij}(0) = \varphi_{ij}, \;\; (x_i, y_j, t) \in \partial\Omega^{n,m} \times [0,T], \;\; (x_i, y_j, t) \in \Omega^{n,m}. \tag{4.2.14}$$

The scheme (4.2.13)-(4.2.13) can be written in the matrix notation

$$U'(t) + A(t)U(t) = F(t), \tag{4.2.15}$$

where, $U(t)$ and $F(t)$ are in $\mathbb{R}^{(n-1)(n-1)}$ and the coefficient matrix $A(t)$ is a pentadiagonal matrix and satisfies $A(t) \in \mathbb{R}^{(n-1)^2} \times \mathbb{R}^{(m-1)^2}$. In what follows $p = (i-1)(n-1) + j$ unless otherwise stated. The entries of $A(t)$ and $F(t)$ are given by

$$\begin{aligned}
A_{pp}(t) &= \frac{2\varepsilon}{\Delta x_i + \Delta x_{i+1}} \left(\frac{1}{\Delta x_{i+1}} + \frac{1}{\Delta x_i} \right) + \frac{2\varepsilon}{\Delta y_j + \Delta y_{j+1}} \left(\frac{1}{\Delta y_{j+1}} + \frac{1}{\Delta y_j} \right) + b_{ij}(t), \\
& \qquad i = 1(1)n-1, \quad j = 1(1)m-1, \\
A_{p,p-1}(t) &= -\frac{2\varepsilon}{\Delta y_j + \Delta y_{j+1}} \left(\frac{1}{\Delta y_j} \right), \qquad i = 1(1)n-1, \quad j = 2(1)m-1, \\
A_{p,p+1}(t) &= -\frac{2\varepsilon}{\Delta y_j + \Delta y_{j+1}} \left(\frac{1}{\Delta y_{j+1}} \right), \qquad i = 1(1)n-1, \quad j = 1(1)m-2, \\
A_{p,p-(n-1)}(t) &= -\frac{2\varepsilon}{\Delta x_i + \Delta x_{i+1}} \left(\frac{1}{\Delta x_i} \right), \qquad i = 2(1)n-1, \quad j = 1(1)m-1, \\
A_{p,p+(n-1)}(t) &= -\frac{2\varepsilon}{\Delta x_i + \Delta x_{i+1}} \left(\frac{1}{\Delta x_{i+1}} \right), \qquad i = 1(1)n-2, \quad j = 1(1)m-1.
\end{aligned}$$

$$\begin{aligned}
F_p(t) &= f_p(t) + \left(\frac{2\varepsilon}{\Delta x_1(\Delta x_1 + \Delta x_2)} \right) u(0, y_1, t) + \left(\frac{2\varepsilon}{\Delta y_1(\Delta y_1 + \Delta y_2)} \right) u(x_1, 0, t), \\
F_p(t) &= f_p(t) + \left(\frac{2\varepsilon}{\Delta x_1(\Delta x_1 + \Delta x_2)} \right) u(0, y_j, t), \qquad j = 2(1)m-2, \\
F_p(t) &= f_p(t) + \left(\frac{2\varepsilon}{\Delta x_1(\Delta x_1 + \Delta x_2)} \right) u(0, y_{n-1}, t) + \left(\frac{2\varepsilon}{\Delta y_m(\Delta y_{m-1} + \Delta y_m)} \right) u(x_1, 1, t), \\
F_p(t) &= f_p(t) + \left(\frac{2\varepsilon}{\Delta y_1(\Delta y_1 + \Delta y_2)} \right) u(x_i, 0, t), \qquad i = 2(1)n-2, \\
F_p(t) &= f_p(t), \qquad i = 2(1)n-2, \quad j = 2(1)m-2, \\
F_p(t) &= f_p(t) + \left(\frac{2\varepsilon}{\Delta y_m(\Delta y_{m-1} + \Delta y_m)} \right) u(x_i, 1, t), \qquad i = 2(1)n-2, \\
F_p(t) &= f_p(t) + \left(\frac{2\varepsilon}{\Delta x_n(\Delta x_{n-1} + \Delta x_n)} \right) u(1, y_{m-1}, t) + \left(\frac{2\varepsilon}{\Delta y_1(\Delta y_1 + \Delta y_2)} \right) u(x_i, 0, t), \\
F_p(t) &= f_p(t) + \left(\frac{2\varepsilon}{\Delta x_n(\Delta x_{n-1} + \Delta x_n)} \right) u(1, y_j, t), \qquad j = 2(1)m-2, \\
F_p(t) &= f_p(t) \left(\frac{2\varepsilon}{\Delta x_n(\Delta x_{n-1} + \Delta x_n)} \right) u(1, y_{m-1}, t) + \left(\frac{2\varepsilon}{\Delta y_m(\Delta y_{m-1} + \Delta y_m)} \right) u(x_{n-1}, 1, t).
\end{aligned}$$

Next we estimate the error associated with this spatial discretization.

4.2.2 Error analysis

Below we analyse the FMFDM for convergence. To be able to carry out the said analysis, the following lemmas are needed. These lemmas have been established from [12].

Lemma 4.2.1. *(Semi-discrete maximum principle) Let* $\xi_{ij}(t)$ *be sufficiently smooth semi-discrete function defined on* $\bar{\Omega}^{n,m} \times [0,T]$*. If* $\xi_{ij}(t)$ *satisfies* $\xi_{0j}(t) \geq 0,\ \xi_{i0}(t) \geq 0,\ \xi_{nj}(t) \geq 0,\ \xi_{im}(t) \geq 0,\ \forall\ (x_i, y_j, t) \in \bar{\Omega}^{m,n} \times [0,T]$ *and* $\mathcal{L}^{n,m}\xi_{ij}(t) > 0,\ \forall\ (x_i, y_j, t) \in \Omega^{n,m} \times [0,T]$*, then* $\xi_{ij}(t) \geq 0,\ \forall\ (x_i, y_j, t) \in \bar{\Omega}^{n,m} \times [0,T]$.

Proof. Let (l, s) be indices such that

$$\xi_{ls}(t) = \min_{(x_i,y_j,t)\in\bar{\Omega}^{n,m}\times[0,T]} \xi_{ls}(t),$$

holds and suppose $\xi_{ls}(t) < 0$. Then $(l,s) \in \{1,2,...,n-1\} \times \{1,2,...,m-1\}$. We see that $(\xi_{ls}(t))_t = 0,\ \xi_{ls}(t) < \xi_{l+1,s}(t),\ \ \xi_{l-1,s}(t) < \xi_{ls}(t),\ \ \xi_{ls}(t) < \xi_{l,s+1}(t),\ \ \xi_{l,s-1}(t) < \xi_{ls}(t)$, thus we have $\mathcal{L}^{n,m}\xi_{ls}(t) < 0$, which is a contradiction. □

Lemma 4.2.2. *The solution* $u_{ij}(t)$ *of the semi-discrete problem (4.2.13)-(4.2.14), satisfies the bound*

$$||u_{ij}(t)|| \leq \beta^{-1} \max_{(x_i,y_j,t)\in\bar{\Omega}^{n,m}\times[0,T]} ||\mathcal{L}^{n,m}u_{ij}(t)|| + \max_{(x_i,y_j,t)\in\bar{\Omega}^{n,m}\times[0,T]} (||\varphi_{ij}||, g_{ij}(t)).$$

Proof. Let $p = \beta^{-1} \max_{(x_i,y_j,t)\in\bar{\Omega}^{n,m}\times(0,T)} ||\mathcal{L}^{n,m}u_{ij}(t)|| + \max_{(x_i,y_j,t)\in\bar{\Omega}^{n,m}\times[0,T]} (||\varphi_{ij}||, g_{ij}(t))$, and define the function $\psi^{\pm}_{ij}(t)$ by

$$\psi^{\pm}_{ij}(t) = p \pm u_{ij}(t).$$

A the boundaries we have

$$\begin{aligned}
\psi^{\pm}_{0j}(t) &= p \pm u_{0j}(t) = p \pm g_{0j}(t) \geq 0, \\
\psi^{\pm}_{nj}(t) &= p \pm u_{nj}(t) = p \pm g_{nj}(t) \geq 0, \\
\psi^{\pm}_{i0}(t) &= p \pm u_{i0}(t) = p \pm g_{i0}(t) \geq 0, \\
\psi^{\pm}_{im}(t) &= p \pm u_{im}(t) = p \pm g_{im}(t) \geq 0,
\end{aligned}$$

and for the domain $0 < i < n,\ 0 < j < m$, we obtain

$$\mathcal{L}^{n,m}\psi^{\pm}_{ij}(t) = -\varepsilon \left[\frac{2}{\Delta x_i + \Delta x_{i+1}} \left(\frac{p \pm u_{i+1,j}(t) - p \pm u_{ij}(t)}{\Delta x_{i+1}} - \frac{p \pm u_{ij}(t) - p \pm u_{i-1,j}(t)}{\Delta x_i} \right) \right.$$

$$
\begin{aligned}
&+\frac{2}{\Delta y_j+\Delta y_{j+1}}\left(\frac{p\pm u_{i,j+1}(t)-p\pm u_{ij}(t)}{\Delta y_{j+1}}-\frac{p\pm u_{ij}(t)-p\pm u_{i,j-1}(t)}{\Delta y_j}\right)\Bigg]\\
&+b_{ij}(t)(p\pm u_{ij}(t))+(p\pm u_{ij}(t))_t\\
=\ & b_{ij}(t)p\pm \mathcal{L}^{n,m}u_{ij}(t)\\
=\ & b_{ij}(t)p\pm f_{ij}(t)\\
=\ & b_{ij}(t)\left[\beta^{-1}\max_{(x_i,y_j,t)\in\Omega^{n,m}\times[0,T]}||\mathcal{L}^{n,m}u_{ij}(t)||+\max_{(x_i,y_j,t)\in\Omega^{n,m}\times[0,T]}(||\varphi_{ij}||,g_{ij}(t))\right]\pm f_{ij}(t)\\
\geq\ & 0,\ \text{since } b_{ij}(t)\geq\beta.
\end{aligned}
$$

Therefore $\psi^{\pm}_{ij}(t)\geq 0$, $\forall\,(x_i,y_j,t)\in\bar{\Omega}^{m,n}\times[0,T]$, and this completes the proof. □

Now we bound the error. For simplicity we assume throughout the analysis that $n=m,\ \sigma=\sigma_x=\sigma_y$. Similar to the exact solution, we write the numerical solution $U_i(t)$ as the sum

$$
U_{ij}(t)=V_{ij}(t)+\sum_{p=1}^{4}W_{p_{ij}}(t)+\sum_{p=1}^{4}Z_{p_{ij}}(t),\ \forall\,(x_i,y_j,t)\in\Omega^n\times(0,T],
$$

representing the regular component, the four edge layer functions and the four corner layer functions respectively. For all $(x_i,y_j,t)\in\bar{\Omega}^n\times(0,T]$, each term satisfies

$$
\begin{aligned}
\mathcal{L}^nV_{ij}(t) &= f(t),\ \ V_{ij}(0)=v_{ij}(0), && (4.2.16)\\
\mathcal{L}^nW_{p_{ij}}(t) &= g_{p_{ij}}(t)+\frac{\partial g_{ij}(t)}{dt},\ \ W_{p_{ij}}(0)=w_{p_{ij}}(0),\ \ p=1,2,3,4, && (4.2.17)\\
\mathcal{L}^nZ_{p_{ij}}(t) &= \tilde{g}_{p_{ij}}+\frac{\partial\tilde{g}_{ij}(t)}{\partial t},\quad Z_{p_{ij}}(0)=z_{p_{ij}}(0),\quad p=1,2,3,4. && (4.2.18)
\end{aligned}
$$

Here $g_{p_{ij}}(t)$ is defined as some boundary conditions located at the four edges of Ω^n and $\tilde{g}_{p_{ij}}$ has the boundary conditions at the four corners.

Error of the regular component

To bound the error of the regular part of the numerical solution we write it as

$$
\begin{aligned}
\mathcal{L}^n(V_{ij}(t)-v_{ij}(t)) &= f_{ij}(t)-\mathcal{L}^nv_{ij}(t)\\
&= (\mathcal{L}-\mathcal{L}^n)\,v_{ij}(t).
\end{aligned}
$$

Note that like the one-dimensional case the error analysis does not include the time derivative term thus we follow [6] to present the analysis. When $x=\sigma_x, 1-\sigma_x$ or $y_j=$

$\sigma_y,\ 1-\sigma_y$, we obtain the estimate

$$\begin{aligned}\mathcal{L}^n(V_{ij}(t)-v_{ij}(t)) &= \Bigg\|-\varepsilon\Bigg[(v_{xx}(t))_{ij}+(v_{yy}(t))_{ij}-\frac{2}{\Delta x_i+\Delta x_{i+1}}\Bigg(\frac{v_{i+1,j}(t)-v_{ij}(t)}{\Delta x_{i+1}} \\ &\quad -\frac{v_{ij}(t)-v_{i-1,j}(t)}{\Delta x_i}\Bigg)-\frac{2}{\Delta y_j+\Delta y_{j+1}}\Bigg(\frac{v_{i,j-1}(t)-v_{ij}(t)}{\Delta y_{j+1}} \\ &\quad -\frac{v_{ij}(t)-v_{i,j-1}(t)}{\Delta y_j}\Bigg)\Bigg]\Bigg\|.\end{aligned}$$

Using appropriate Taylor series expansions transforms the error into

$$\begin{aligned}\mathcal{L}^n(V_{ij}(t)-v_{ij}(t)) &\le \left(\frac{1}{\Delta x_{i+1}}\int_{x_i}^{x_{i+1}}(x_{i+1}-s)^2ds+\frac{1}{\Delta x_i}\int_{x_i}^{x_{i-1}}(s-x_{i-1})^2ds\right) \\ &\quad \times\frac{\varepsilon||(v_{xxx}(t))_{ij}||}{\Delta x_i+\Delta x_{i+1}}+\frac{\varepsilon||(v_{yyy}(t))_{ij}||}{\Delta y_j+\Delta y_{j+1}} \\ &\quad \times\left(\frac{1}{\Delta y_{j+1}}\int_{y_j}^{y_{j+1}}(y_{j+1}-s)^2ds+\frac{1}{\Delta y_j}\int_{y_j}^{y_{j-1}}(s-y_{j-1})^2ds\right).\end{aligned}$$

Evaluating the integral gives

$$\begin{aligned}\mathcal{L}^n(V_{ij}(t)-v_{ij}(t)) &\le \frac{\varepsilon||(v_{xxx}(t))_{ij}||}{\Delta x_i+\Delta x_{i+1}}\left(\frac{1}{3\Delta x_{i+1}}(x_{i+1}-x_i)^3-\frac{1}{3\Delta x_i}(x_i-x_{i-1})^3\right) \\ &\quad +\frac{\varepsilon||(v_{yyy}(t))_{ij}||}{\Delta y_j+\Delta y_{j+1}}\left(\frac{1}{3\Delta y_{j+1}}(y_{j+1}-y_j)^3-\frac{1}{3\Delta y_j}(y_j-y_{j-1})^3\right) \\ &= \frac{\varepsilon||(v_{xxx}(t))_{ij}||}{3(\Delta x_i+\Delta x_{i+1})}\left((x_{i+1}-x_i)^2-(x_i-x_{i-1})^2\right)+\frac{\varepsilon||(v_{yyy}(t))_{ij}||}{3(\Delta y_j+\Delta y_{j+1})} \\ &\quad \times\left((y_{j+1}-y_j)^2-(y_j-y_{j-1})^2\right) \\ &\le \frac{\varepsilon}{3}(x_{i+1}-x_{i-1})||(v_{xxx}(t))_{ij}||+\frac{\varepsilon}{3}(y_{j+1}-y_{j-1})||(v_{yyy}(t))_{ij}||.\end{aligned}$$

When $(x_i, y_j)\in(\sigma,1-\sigma)$, we have the truncation error as

$$\begin{aligned}\mathcal{L}^n(V_{ij}(t)-v_{ij}(t)) &= \Bigg\|-\varepsilon\Bigg[(v_{xx}(t))_{ij}+(v_{yy}(t))_{ij}-\left(\frac{v_{i+1,j}(t)-2v_{ij}(t)+v_{i-1,j}(t)}{(\Delta x)_i^2}\right) \\ &\quad -\left(\frac{v_{i,j-1}(t)-2v_{ij}(t)+v_{i,j-1}(t)}{(\Delta y)_j^2}\right)\Bigg]\Bigg\| \\ &\le \frac{\varepsilon}{12}(\Delta x)_i^2||(v_{xxxx}(t))_{ij}||+\frac{\varepsilon}{12}(\Delta y)_j^2||(v_{yyyy}(t))_{ij}||.\end{aligned}$$

Now putting these two truncation errors together, we obtain

$$||\mathcal{L}^n(V_{ij}(t)-v_{ij}(t))|| \le \begin{cases} C\varepsilon((x_{i+1}-x_{i-1})||(v_{xxx}(t))_{ij}||+(y_{j+1}-y_{j-1})||(v_{yyy}(t))_{ij}||), \\ \text{if } x_i=\sigma_x, 1-\sigma_x \text{ or } y_j=\sigma_y,\ 1-\sigma_y, \\ C\varepsilon((\Delta x)_i^2||(v_{xxxx}(t))_{ij}||+(\Delta y)_j^2||(v_{yyyy}(t))_{ij}||), \quad \text{otherwise.}\end{cases}$$

Applying the bound (4.1.4) gives

$$||\mathcal{L}^n(V_{ij}(t)-v_{ij}(t))|| \leq \begin{cases} C\varepsilon((x_{i+1}-x_{i-1})+(y_{j+1}-y_{j-1})), \\ \text{if } x_i=\sigma_x, 1-\sigma_x \text{ or } y_j=\sigma_y,\ 1-\sigma_y, \\ C\varepsilon((\Delta x)_i^2+(\Delta y)_j^2), \quad \text{otherwise,} \end{cases}$$

and from the inequality (4.1.4) we obtain

$$\mathcal{L}^n(V_{ij}(t)-v_{ij}(t)) \leq \begin{cases} C\varepsilon n^{-1}, \quad \text{if } x_i=\sigma_x, 1-\sigma_x \text{ or } y_j=\sigma_y,\ 1-\sigma_y, \\ c\varepsilon n^{-2}, \quad \text{otherwise.} \end{cases}$$

Now we follow [6] to define the barrier function

$$\Phi(x_i,y_j,t)=\frac{\sigma_x\sigma_y}{\varepsilon}n^{-2}(\theta(x_i)+\theta(y_j))+cn^{-2},$$

where $\theta(\gamma)$ is as defined in Chapter 2. From the value of the transition point we have

$$0\leq\Phi(x_i,y_j,t)\leq C(n^{-1}\ln n)^2,$$

and for $\mathcal{L}^n$ on the barrier function when $x_i=\sigma_x,\ 1-\sigma_x$ or $y_j=\sigma_y,\ 1-\sigma_y$, we obtain

$$\begin{aligned}\mathcal{L}^n\Phi(x_i,y_j,t) &= \Phi_t-\varepsilon\left[\frac{\sigma_x\sigma_y}{\varepsilon}n^{-2}\left(-\frac{n}{\sigma_x}-\frac{n}{\sigma_y}\right)\right]+b_{ij}(t)\Phi \\ &= Cn^{-1}(\sigma_x+\sigma_y)+b_{ij}(t)\Phi.\end{aligned}$$

Otherwise, we have $\mathcal{L}^n\Phi(x_i,y_j,t)=b_{ij}(t)\Phi$.

$$\mathcal{L}^n\Phi(x_i,y_j)\geq\begin{cases} Cn^{-1}(\sigma_x+\sigma_y)+b_{ij}(t)\Phi, & \text{if } x=\sigma_x, 1-\sigma_x \text{ or } y_j=\sigma_y,\ 1-\sigma_y, \\ b_{ij}(t)\Phi, & \text{otherwise.}\end{cases}$$

Using the barrier function $\Psi^{\pm}(x_i,y_j,t)=\Phi(x_i,y_j,t)\pm(V(t)-v(t))(x_i,y_j)$, it satisfies

$$\begin{aligned}\Psi^{\pm}(x_i,y_j,t) &\geq 0,\ \forall\,(x_i,y_j,t)\in\partial\Omega^n\times[0,T] \text{ and} \\ \mathcal{L}^n\Psi^{\pm}(x_i,y_j,t) &\geq 0,\ \forall\ (x_i,y_j,t)\in\Omega^n\times[0,T].\end{aligned}$$

Thus from Lemma 4.2.2 it follows that $\Psi^{\pm}(x_i,y_j,t)\geq 0,\ (x_i,y_j,t)\in\bar{\Omega}^n\times[0,T]$, therefore

$$||V_{ij}(t)-v_{ij}(t)||\leq\Phi(x_i,y_j,t)\leq C(n^{-1}\ln n)^2.$$

Proposition 4.2.1. *The error of the regular component satisfies*

$$||V_{ij}(t)-v_{ij}(t)||\leq C(n^{-1}\ln n)^2.$$

Error around the edges of the domain

To bound the error around the edge $W_{1ij}(t)$, we consider two different regions in the i domain; the region $0 < i < n/4,\ 0 < j < n,\ 0 < t \leq T$ and the region $n/4 \leq i \leq n,\ 0 \leq j \leq n,\ 0 < t \leq T$. To bound the error in the region $0 < i < n/4,\ 0 < j < n,\ 0 < t \leq T$ we follow the same argument we used to bound the error in the regular component since the mesh is very fine. Now we estimate the error as follows

$$\mathcal{L}^n(W_{1ij}(t) - w_{1ij}(t)) = -\varepsilon\left[\left(\frac{\partial^2 x}{\partial x^2} - \delta_{xx}\right) + \left(\frac{\partial^2 y}{\partial y^2} - \delta_{yy}\right)\right] w_{1ij}(t).$$

Using appropriate Taylor series expansions and following the same calculations we did for the regular component, we obtain the estimate

$$\mathcal{L}^n(W_{1ij}(t) - w_{1ij}(t)) \leq \begin{cases} C\varepsilon\left[(\Delta x_i)^2 ||(w_{1xxxx}(t))_{ij}|| + (y_{j+1} - y_{j-1})||(w_{1yyy}(t))_{ij}||\right], \\ j = n/4, 3n/4, \\ C\varepsilon\left[(\Delta x_i)^2 ||(w_{1xxxx}(t))_{ij}|| + (\Delta y_j)^2 ||(w_{1yyyy}(t))_{ij}||\right], \\ \text{otherwise.} \end{cases}$$

From the bound (4.1.5) we have

$$||\mathcal{L}^n(W_{1ij}(t) - w_{1ij}(t))|| \leq \begin{cases} C(\Delta x_i)^2\varepsilon^{-1} + C\varepsilon\Delta y_j, & j = n/4, 3n/4, \\ C(\Delta x_i)^2\varepsilon^{-1} + C\varepsilon(\Delta y_j)^2, & \text{otherwise.} \end{cases}$$

$$||\mathcal{L}^n(W_{1ij}(t) - w_{1ij}(t))|| \leq \begin{cases} C(n^{-1}\ln n)^2 + C\varepsilon n^{-1}, & j = n/4, 3n/4, \\ C(n^{-1}\ln n)^2, & \text{otherwise.} \end{cases}$$

We employ the barrier function

$$\Phi(x_i, y_j, t) = C\frac{\sigma_y}{\sqrt{\varepsilon}} n^{-2}\theta(y_j) + C(n^{-1}\ln)^2,$$

where $\theta(y_j)$ is as defined in Chapter 2. At the transition point the barrier function yields

$$0 \leq \Phi(x_i, y_j) \leq C(n^{-1}\ln n)^2,$$

and the semi-discrete operator on the barrier function when $y_j = \sigma_y,\ 1 - \sigma_y$, is

$$\begin{aligned} \mathcal{L}^n\Phi(x_i, y_j, t) &= \Phi_t - \varepsilon\left[\frac{\sigma_y}{\sqrt{\varepsilon}} n^{-2}\left(-\frac{n}{\sigma_y}\right)\right] + b_{ij}(t)\Phi \\ &= Cn^{-1}\sqrt{\varepsilon} + b_{ij}(t)\Phi. \end{aligned}$$

Otherwise, we have $\mathcal{L}^n\Phi(x_i, y_j, t) = b_{ij}(t)\Phi$. Collecting these results together gives

$$\mathcal{L}^n\Phi(x_i, y_j, t) \geq \begin{cases} C\sqrt{\varepsilon}n^{-1} + b_{ij}(t)\Phi, & j = n/4,\ 3n/4, \\ b_{ij}(t)\Phi, & \text{otherwise.} \end{cases}$$

From the maximum principle

$$\begin{aligned} \Phi_{0j}(t) &= (Cn^{-1}\ln n)^2 \geq 0, \\ \Phi_{n/4j}(t) &= (Cn^{-1}\ln n)^2 \geq 0, \forall\, 0 \leq i \leq n/4,\ 0 < j < n,\ 0 \leq t \leq T, \\ \mathcal{L}^n\Phi_{ij}(t) &= (Cn^{-1}\ln n)^2 \geq 0,\ \ \forall\, 0 < i < n/4,\ 0 < j < n,\ 0 \leq t \leq T. \\ \therefore \Phi_{ij}(t) &= (Cn^{-1}\ln n)^2 \geq 0, \end{aligned}$$

holds and thus we obtain

$$||W_{1ij}(t) - w_{1ij}(t)|| \leq C(n^{-1}\ln n)^2,\ \forall\, 0 < i < n/4,\ 0 < j < n,\ 0 \leq t \leq T. \quad (4.2.19)$$

To bound the error in the region $n/4 \leq i \leq n,\ 0 \leq j \leq n,\ 0 \leq t \leq T$, we follow [7] to define the barrier function

$$B_{w1i}(t) = \begin{cases} \prod_{s=1}^{i}(1 + \Delta x_s\sqrt{\beta/\varepsilon})^{-1}, & i \neq 0, \\ 1, i = 0, \end{cases},$$

which is a first order Taylor series expansion of the boundary layer terms $\exp(-\sqrt{\beta/\varepsilon}x)$. For all i we have the inequality

$$\exp(-\sqrt{\beta/\varepsilon}x_i) = \prod_{s=1}^{i}\exp(-\sqrt{\beta/\varepsilon}\Delta x_s).$$

Since for each s such that $1 \leq s \leq i$,

$$\exp(-\Delta x_s\sqrt{\beta/\varepsilon}) \leq \frac{1}{1 + \Delta x_s\sqrt{\beta/\varepsilon}},$$

we have

$$\prod_{s=1}^{i}\exp(-\sqrt{\beta/\varepsilon}\Delta x_s) \leq B_{w1i}(t).$$

For $\sigma < 0.25$ and $n/4 \leq i \leq n$, we have

$$B_{w1i}(t) \leq B_{w1n/4}(t) = \left(1 + \frac{8\ln n}{n}\right)^{-n/4} \leq Cn^{-2}. \quad (4.2.20)$$

For more details of the inequality (4.2.20) refer to page 32 of [34].

also holds Now for the semi-discrete operator on the barrier function $B_i(t)$, we have

$$
\begin{aligned}
D_x^+ B_{w1i}(t) &= \frac{\prod_{s=1}^{i+1}\left(1+\Delta x_s\sqrt{\beta/\varepsilon}\right)^{-1} - \prod_{s=1}^{i}\left(1+\Delta x_s\sqrt{\beta/\varepsilon}\right)^{-1}}{\Delta x_{i+1}} \\
&= -\sqrt{\beta/\varepsilon}(1+\Delta x_{i+1}\sqrt{\beta/\varepsilon})^{-1}\prod_{s=1}^{i}(1+\Delta x_s\sqrt{\beta/\varepsilon})^{-1}
\end{aligned}
$$

$$
\begin{aligned}
D_x^- B_{w1i}(t) &= \frac{\prod_{s=1}^{i}\left(1+\Delta x_s\sqrt{\beta/\varepsilon}\right)^{-1} - \prod_{s=1}^{i-1}\left(1+\Delta x_s\sqrt{\beta/\varepsilon}\right)^{-1}}{\Delta x_{i}} \\
&= -\sqrt{\beta/\varepsilon}\prod_{s=1}^{i}(1+\Delta x_s\sqrt{\beta/\varepsilon})^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}^n B_{w1i}(t) &= (B_{w1i}(t))_t - \varepsilon(\delta_x^2+\delta_y^2)B_{w1i}(t) + b_{ij}(t)B_{w1i}(t) \\
&= -\frac{2\varepsilon}{\Delta x_i+\Delta x_{i+1}}\left(-\sqrt{\beta/\varepsilon}(1+\Delta x_{i+1}\sqrt{\beta/\varepsilon})^{-1}+\sqrt{\beta/\varepsilon}+b_{ij}(t)\right)B_{w1i}(t) \\
&= \left(b_{ij}(t) - \frac{2\beta}{\Delta x_i+\Delta x_{i+1}(1+\Delta x_{i+1}\sqrt{\beta/\varepsilon})}\right)B_{w1i}(t) \\
&\geq (b_{ij}(t)-C\beta)B_{w1i}(t). \qquad (4.2.21)
\end{aligned}
$$

At the boundary of the domain $n/4 \leq i \leq n,\ 0 \leq j \leq n,\ 0 \leq t \leq T$, the numerical solution is equal to the exact solution. Thus from the bound (4.1.5), we have

$$
W_{1ij}(t) = w_{1ij}(t) \leq C\exp(-\sqrt{\beta/\varepsilon}x_i) \leq CB_{w1i}(t),\ \ \forall\ (x_i,y_j)\in\partial\Omega^n\ ,\ t\in\ (0,T].
$$

Now when we consider the mesh points $0 < i, j < n$, from equation (4.2.17), we have

$$
\mathcal{L}^n W_{1ij}(t) = g_{1ij}(t) + \frac{\partial g_{ij}(t)}{\partial t}, \qquad (4.2.22)
$$

and from the inequality (4.2.21)

$$
\mathcal{L}^n B_{w1i}(t) \geq (b_{ij}(t)-C\beta)B_{w1i}(t) \geq \mathcal{L}^n W_{1ij}(t)
$$

also holds. It follows that $B_{w1i}(t)$ is a barrier function for $W_{1ij}(t)$, thus we have

$$
W_{1ij}(t) \leq B_{w1i}(t). \qquad (4.2.23)
$$

Using the bound (4.1.5), and the inequality (4.2.20), we obtain the estimate

$$\begin{aligned} ||W_{1ij}(t) - w_{1ij}(t)|| &\leq ||W_{1ij}(t)|| + ||w_{1ij}(t)|| \\ &\leq CB_{w1i} \leq Cn^{-2}, \ \forall\ 0 < i < n/4, 0 < j < n, 0 < t \leq T. \end{aligned}$$

Combining the results in each domain yields the bound

$$||(W_1(t) - w_1(t))(x_i, y_j)|| \leq C(n^{-1}\ln n)^2, \quad (x_i, y_j) \in \Omega^n, \ \ t \in (0, T]. \tag{4.2.24}$$

The error bound around the other three edges can be obtained in an analogous manner.

Proposition 4.2.2. *For $p = 1, 2, 3, 4$, the error bound associated with the four edges satisfy*

$$||(W_{p_{ij}}(t) - w_{p_{ij}}(t))|| \leq C(n^{-1}\ln n)^2 \quad (x_i, y_j) \in \Omega^n, \ \ t \in (0, T].$$

Where $w_{p_{ij}}(t)$ and $W_{p_{ij}}(t)$ are the exact and the numerical solutions respectively.

Error around the corners of the domain

Similar to the estimate of the edge $W_{1ij}(t)$, we bound the error in the domain $n/4 \leq i, j \leq n$, and the domain $\Omega^n \setminus n/4 \leq i, j \leq n$. In the domain $n/4 \leq i, j \leq n$, the error is given as

$$\mathcal{L}^n(Z_{1ij}(t) - z_{1ij}(t)) = -\varepsilon\left[\left(\frac{\partial^2 x}{\partial x^2} - \delta_{xx}\right) + \left(\frac{\partial^2 y}{\partial y^2} - \delta_{yy}\right)\right] z_{1ij}(t).$$

Using appropriate Taylor series expansions and the bound (4.1.9) gives

$$\begin{aligned} ||\mathcal{L}^n(Z_{1ij}(t) - z_{1ij}(t))|| &\leq C\varepsilon(\Delta x)_i^2(|||(z_{1xxxx}(t))_{ij}|| + ||(z_{1yyyy}(t))_{ij}|| \\ &\leq C\varepsilon(\sigma n^{-1})^2\varepsilon^{-2} \\ &\leq (Cn^{-1}\ln n)^2. \end{aligned}$$

Following [19], we define the barrier function

$$\Phi(x_i, y_j, t) = C(n^{-1}\ln n)^2.$$

Which satisfies the inequalities

$$\begin{aligned} \Phi_{n/4n/4}(t) &= (Cn^{-1}\ln n)^2 \geq 0, \\ \Phi_{nn}(t) &= (Cn^{-1}\ln n)^2 \geq 0, \forall\, n/4 \leq i, j \leq n, \ t \in [0, T], \\ \mathcal{L}^n\Phi_{ij}(t) &= (Cn^{-1}\ln n)^2 \geq 0, \ \forall\, n/4 < i, j < n, \ t \in [0, T]. \end{aligned}$$

Thus from the semi-discrete maximum principle 4.2.1,

$$||Z_1(t) - z_1(t))(x_i, y_j)|| \leq (Cn^{-1}\ln n)^2,$$

follows. Now to bound the error in the domain $\Omega^n \setminus n/4 \leq i, j \leq n$, we know that the numerical solution is equal to the exact solution at the boundaries thus we have

$$||Z_1(t)(x_i, y_j)|| = ||z_1(t)(x_i, y_j)|| \leq C \min\left\{\exp\left(-\sqrt{\beta/\varepsilon}x_i\right), \exp\left(-\sqrt{\beta/\varepsilon}y_j\right)\right\}.$$

When $(x_i, y_j) \in \Omega^n \setminus n/4 \leq i, j \leq n,\ \ t \in (0, T]$, we use the barrier function

$$B_{w2j}(t) = \begin{cases} \prod_{s=1}^{j}(1 + \Delta y_s\sqrt{\beta/\varepsilon})^{-1}, & j \neq 0, \\ 1, j = 0. \end{cases}$$

together with $B_{w1i}(t)$. Now following the same reasoning like before, the error around $(0, 0)$ satisfies

$$||(Z_{1ij}(t) - z_{1ij}(t))|| \leq C \min\{B_{w1i}, B_{w2j}\}.$$

From the inequality 4.2.20 we have

$$||Z_1(t)(x_i, y_j) - z_1(t)(x_i, y_j)|| \leq Cn^{-2}.$$

Adding the preceding error yields

$$||Z_1(t)(x_i, y_j) - z_1(t)(x_i, y_j)|| \leq C(n^{-1}\ln n)^2.$$

Similar bounds holds for the other three corner layer functions.

Proposition 4.2.3. *Let $z_p(t)$ and $Z_p(t)$ be the exact and the numerical solutions respectively then for all $(x_i, y_j, t) \in \Omega^n \times (0, T]$, we have*

$$||(Z_{p_{ij}}(t) - z_{p_{ij}}(t))|| \leq C(n^{-1}\ln n)^2. \tag{4.2.25}$$

Lemma 4.2.3. *From propositions 4.2.1-4.2.2 and 4.2.3 the error of the spatial semi-descretization with the FMFDM satisfies*

$$||U_{ij}(t) - u_{ij}(t)|| \leq C(n^{-1}\ln n)^2.$$

Next we discretize in space with the FOFDM.

4.3 Spatial discretization with the FOFDM

Here we descretize problem (4.1.1)-(4.1.2) in space via a fitted operator finite difference scheme.

4.3.1 The FOFDM

We consider the following partitions of the interval $[0, 1]$:

$$x_0 = 0, \;\; x_i = x_0 + i\Delta x, \;\; i = 1(1)n-1, \;\; \Delta x = x_i - x_{i-1}, \;\; x_n = 1,$$

$$y_0 = 0, \;\; y_j = x_0 + j\Delta y, \;\; j = 1(1)m-1, \;\; \Delta y = y_j - y_{j-1}, \;\; y_m = 1.$$

Note that the tensor product of these two partitions gives the mesh grid on the entire domain Ω. Using the theory of difference schemes and the notations from 4.2.1, we design the scheme as follows:

$$\begin{aligned}\mathcal{L}^{\Delta x,\Delta y}U_{ij}(t) \equiv & \frac{dU_{ij}(t)}{dt} - \varepsilon\left(\frac{U_{i+1,j}(t) - 2U_{ij}(t) + U_{i-1,j}(t)}{(\phi_{ij})^2(\varepsilon, \Delta x, t)} + \frac{U_{i,j+1}(t) - 2U_{ij}(t) + U_{i,j-1}(t)}{(\phi_{ij})^2(\varepsilon, \Delta y, t)}\right) \\ & + b_{ij}(t)U_{ij}(t) = f_{ij}(t), \quad i = 1(1)n-1, \; j = 1(1)m-1, \end{aligned} \tag{4.3.26}$$

subject to the semi-discrete boundary and initial conditions

$$\begin{aligned} U_{0j}(t) &= g_{0j}(t), \; U_{nj}(t) = g_{nj}(t), \; U_{i0}(t) = g_{0j}(t), \; U_{im}(t) = g_{im}(t), \;\; (x_i, y_j, t) \in \bar{\Omega}^{n,m}, \\ U_{ij}(0) &= \varphi_{ij}, \;\; (x_i, y_j) \in \bar{\Omega}^n. \end{aligned} \tag{4.3.27}$$

The denominator functions $\phi_{ij}(\varepsilon, \Delta y, t)$ and $\phi_{ij}(\varepsilon, \Delta y, t)$, are given as

$$\phi_{ij}(\varepsilon, \Delta x, t) = \frac{2}{\rho_{ij}} \sinh\left(\frac{\rho_{ij}\Delta x}{2}\right),$$

$$\phi_{ij}(\varepsilon, \Delta y, t) = \frac{2}{\rho_{ij}} \sinh\left(\frac{\rho_{ij}\Delta y}{2}\right),$$

with $\rho_{ij} = \sqrt{\frac{b_{ij}(t)}{\varepsilon}}$. We represent the difference scheme (4.3.26)-(4.3.27) in the matrix notation

$$U'(t) + A(t)U(t) = F(t), \tag{4.3.28}$$

where $A(t) \in \mathbb{R}^{(n-1)(m-1)} \times \mathbb{R}^{(n-1)(m-1)}$ and $F(t) \in \mathbb{R}^{(n-1)(m-1)}$ and have the entries given below.

$$\begin{aligned}
A_{pp}(t) &= \frac{2\varepsilon}{(\phi_{ij})^2(\varepsilon,\Delta x,t)} + \frac{2\varepsilon}{(\phi_{ij})^2(\varepsilon,\Delta y,t)} + b_{ij}(t), \; i=1(1)n-1, \;\; j=1(1)m-1,\\
A_{p,p+1}(t) &= -\frac{\varepsilon}{(\phi_{ij})^2(\varepsilon,\Delta y,t)}, \qquad i=1(1)n-1, \;\; j=1(1)m-2,\\
A_{p,p-1}(t) &= -\frac{\varepsilon}{(\phi_{ij})^2(\varepsilon,\Delta y,t)}, \qquad i=1(1)n-1, \;\; j=2(1)m-1,\\
A_{p,(n-1)+p}(t) &= -\frac{\varepsilon}{(\phi_{ij})^2(\varepsilon,\Delta x,t)}, \qquad i=1(1)n-2, \;\; j=1(1)m-1,\\
A_{p,p-(n-1)}(t) &= -\frac{\varepsilon}{(\phi_{ij})^2(\varepsilon,\Delta x,t)}, \qquad i=2(1)n-1, \;\; j=1(1)m-1.
\end{aligned}$$

$$\begin{aligned}
F_p(t) &= f_p(t) + \left(\frac{\varepsilon}{(\phi_{11})^2(\varepsilon,\Delta x,t)}\right) u(0,y_1,t) + \left(\frac{\varepsilon}{(\phi_{11})^2(\varepsilon,\Delta y,t)} +\right) u(x_1,0,t),\\
F_p(t) &= f_p(t) + \left(\frac{\varepsilon}{(\phi_{1j})^2(\varepsilon,\Delta x,t)}\right) u(0,y_j,t), \qquad j=2(1)m-2,\\
F_p(t) &= f_p(t) + \left(\frac{\varepsilon}{(\phi_{1,m-1})^2(\varepsilon,\Delta x,t)}\right) u(0,y_{m-1},t) + \left(\frac{\varepsilon}{(\phi_{1,m-1})^2(\varepsilon,\Delta y,t)}\right) u(x_1,1,t),\\
F_p(t) &= f_p(t) + \left(\frac{\varepsilon}{(\phi_{i1})^2(\varepsilon,\Delta y,t)}\right) u(x_i,0,t), \qquad i=2(1)n-2,\\
F_p(t) &= f_p(t), \qquad i=2(1)n-2, \; j=2(1)m-2,\\
F_p(t) &= f_p(t) + \left(\frac{\varepsilon}{(\phi_{i,m-1})^2(\varepsilon,\Delta y,t)}\right) u(x_1,1,t), \qquad i=2(1)n-2,\\
F_p(t) &= f_p(t) + \left(\frac{\varepsilon}{(\phi_{n-1,1})^2(\varepsilon,\Delta x,t)}\right) u(1,y_1,t) + \left(\frac{\varepsilon}{(\phi_{n-1,1})^2(\varepsilon,\Delta y,t)}\right) u(x_{n-1},0,t),\\
F_p(t) &= f_p(t) + \left(\frac{\varepsilon}{(\phi_{n-1,j})^2(\varepsilon,\Delta x,t)}\right) u(1,y_j,t), \qquad j=2(1)m-2,\\
F_p(t) &= f_p(t) \left(\frac{\varepsilon}{(\phi_{n-1,m-1})^2(\varepsilon,\Delta x,t)}\right) u(1,ym-1,t) + \left(\frac{\varepsilon}{(\phi_{n-1,m-1})^2(\varepsilon,\Delta y,t)}\right) u(x_{n-1},1,t).
\end{aligned}$$

4.3.2 Error analysis

Below we provide the theoretical error analysis of the scheme (4.3.26)-(4.3.27). The truncation error is given by

$$\begin{aligned}
\mathcal{L}^{\Delta x,\Delta y}(U_{ij}(t) - u_{ij}(t)) &= f_{ij}(t) - \mathcal{L}^{\Delta x,\Delta y} u_{ij}(t)\\
&= \left(\mathcal{L} - \mathcal{L}^{\Delta x,\Delta y}\right) u_{ij}(t)
\end{aligned}$$

$$
\begin{aligned}
&= -\varepsilon(u_{xx}(t))_{ij} - \varepsilon(u_{yy}(t))_{ij} + \varepsilon\left(\frac{u_{i+1,j}(t) - 2u_{ij}(t) + u_{i-1,j}(t)}{(\phi_{ij})^2(\varepsilon, \Delta x, t)}\right) \\
&\quad + \varepsilon\left(\frac{u_{i,j+1}(t) - 2u_{ij}(t) + u_{i,j-1}(t)}{(\phi_{ij})^2(\varepsilon, \Delta y, t)}\right)
\end{aligned}
$$

Using Taylor series expansions of the terms $u_{i+1,j}(t),\ u_{i-1,j}(t),\ u_{i,j+1}(t),\ u_{i,j-1}(t)$ gives

$$
\begin{aligned}
\mathcal{L}^{\Delta x,\Delta y}\left(U_{ij}(t) - u_{ij}(t)\right) \leq\ & -\varepsilon(u_{xx}(t))_{ij} - \varepsilon(u_{yy}(t))_{ij} + \frac{\varepsilon}{(\phi_{ij})^2(\varepsilon, \Delta x, t)} \\
& \times\left(\Delta x^2(u_{xx}(t))_{ij} + \frac{\Delta x^4}{12}(u_{xxxx}(t))_{ij}\xi_1\right), \quad \xi_1 \in (u_{i+1,j}(t), u_{i-1,j}(t)) \\
& + \frac{\varepsilon}{(\phi_{ij})^2(\varepsilon, \Delta y, t)}\left(\Delta y^2(u_{yy}(t))_{ij} + \frac{\Delta y^4}{12}(u_{yyyy}(t))_{ij}\xi_2\right), \\
& \xi_2 \in (u_{i,j+1}(t) + u_{i,j-1}(t)).
\end{aligned}
$$

Again using Taylor series expansions of ϕ_{ij} gives

$$
\begin{aligned}
\mathcal{L}^{\Delta x,\Delta y}\left(U_{ij}(t) - u_{ij}(t)\right) \leq\ & -\varepsilon u_{xx}(t) - \varepsilon u_{yy}(t) + \left(\frac{\varepsilon}{\Delta x^2} - \frac{b_{ij}(t)}{12} + \frac{(b_{ij}(t))^2\Delta x^2}{240\varepsilon}\right) \\
& \left(\Delta x^2(u_{xx}(t))_{ij} + \frac{\Delta x^4}{12}(u_{xxxx}(t))_{ij}\xi_1\right) \\
& + \left(\frac{\varepsilon}{\Delta y^2} - \frac{b_{ij}(t)}{12} + \frac{(b_{ij}(t))^2\Delta y^2}{240\varepsilon}\right) \\
& \times\left(\Delta y^2(u_{yy}(t))_{ij} + \frac{\Delta y^4}{12}(u_{yyyy}(t))_{ij}\xi_2\right) \\
=\ & \varepsilon\frac{\Delta x^2}{12}(u_{xxxx}(t))_{ij}\xi_1 - \frac{b_{ij}(t)\Delta x^2}{12}(u_{xx}(t))_{ij} \\
& - \frac{b_{ij}(t)\Delta x^4}{144}(u_{xxxx}(t))_{ij}\xi_1 + \frac{(b_{ij}(t))^2\Delta x^4}{240\varepsilon}(u_{xx}(t))_{ij} \\
& + \varepsilon\frac{(b_{ij}(t))^2\Delta x^6}{2880\varepsilon}(u_{xxxx}(t))_{ij}\xi_1 + \varepsilon\frac{\Delta y^2}{12}(u_{yyyy}(t))_{ij}\xi_2 \\
& - \varepsilon\frac{b_{ij}(t)\Delta y^2}{12}(u_{yy}(t))_{ij} - \varepsilon\frac{b_{ij}(t)\Delta y^4}{144}(u_{yyyy}(t))_{ij}\xi_2 \\
& + \varepsilon\frac{b_{ij}(t)^2\Delta y^4}{240\varepsilon}(u_{yy}(t))_{ij} + \varepsilon\frac{(b_{ij}(t))^2\Delta y^6}{2880\varepsilon}(u_{yyyy}(t))_{ij}\xi_2
\end{aligned}
$$

Applying the bounds on the derivatives and the use of Lemma2.3.1 results in

$$
\begin{aligned}
\mathcal{L}^{\Delta x,\Delta y}(U_{ij}(t) - u_{ij}(t)) =\ & \left(\frac{\varepsilon}{12} - \frac{b_{ij}(t)}{12}\right)\Delta x^2 - \left(\frac{b_{ij}(t)}{144} - \frac{(b_{ij}(t))^2}{240\varepsilon}\right)\Delta x^2 + \left(\frac{(b_{ij}(t))^2}{2880\varepsilon}\right)\Delta x^2 \\
& + \left(\frac{\varepsilon}{12} - \frac{b_{ij}(t)}{12}\right)\Delta y^2 - \left(\frac{b_{ij}(t)}{144} - \frac{(b_{ij}(t))^2}{240\varepsilon}\right)\Delta y^2 + \left(\frac{(b_{ij}(t))^2}{2880\varepsilon}\right)\Delta y^2 \\
\leq\ & C(\Delta x^2 + \Delta y^2).
\end{aligned}
$$

Invoking Lemma 4.2.2 yields

$$||U_{ij}(t) - u_{ij}(t)|| \leq C(\Delta x^2 + \Delta y^2).$$

Lemma 4.3.1. *Let $u_i(t)$ be the exact solution of (4.1.1)-(4.1.2) and $U_{ij}(t)$ the solution of (4.3.26)-(4.3.27) at $x = x_i$ and $y = y_j$. Then we have*

$$\max_{0<\varepsilon\leq 1} \max_{0\leq i\leq n; 0\leq j\leq m} |U_{ij}(t) - u_{ij}(t)| \leq C(\Delta x^2 + \Delta y^2),$$

where C is a constant independent of $\Delta x, \Delta y$ and ε.

In the next section we perform the second stage of the method of lines.

4.4 Time discretization

Like the early chapter we discretize the IVPs (4.2.14)-(4.2.14) and (4.3.26)-(4.3.27) with the backward Euler method on a uniform mesh. Now we perform the time discretization as follows:

$$\frac{U^k - U^{k-1}}{\tau} + A(t_k)U^k = F(t_k), \quad \text{for } k = 1, ..., K, \tag{4.4.29}$$

with initial condition $U(0) = \varphi$. Rearranging equation (4.4.29) gives the approximate solution

$$U^k = (\mathbf{I} + \tau A(t_k))^{-1} (\tau F(t_k) + U^{k-1}).$$

Theorem 4.4.1 below gives the summary of the work in this chapter.

Theorem 4.4.1. *Let $u \in \mathcal{C}^{4,2}(\bar{Q})$ be the exact solution of the continuous problem (4.1.1)-(4.1.2) and U_{ij}^k be the numerical solution obtained via the FMDML (4.2.13)-(4.2.14) along with (4.4.29) or the FOFDML (4.3.26)-(4.3.27) along with (4.4.29). Then the errors of these methods are as follows:*

$$\sup_{0<\varepsilon\leq 1} \max_{0\leq i\leq n; 0\leq k\leq K} ||U_{ij}^k - u_{ij}^k|| \leq C((n^{-1}\ln n)^2 + \tau), \quad \textit{for the FMFDML}$$

and

$$\sup_{0<\varepsilon\leq 1} \max_{0\leq i\leq n; 0\leq j\leq m; 0\leq k\leq K} ||U_{ij}^k - u_{ij}^k|| \leq C(\Delta x^2 + \Delta y^2 + \tau), \quad \textit{for the FOFDML.}$$

Next, we illustrate practically the theoretical estimates with an example.

4.5 Numerical example

Here we test the performance of the numerical method with the example below. We compute the maximum pointwise error and the numerical rate of convergence. For simplicity we use $n = m$ in all computations. We use the double mesh principle to estimate the error because the exact solution is not available. Thus we calculate the error using the formula

$$E^{\varepsilon}_{n,\tau} = \max_{0\leq i,j\leq n;0\leq k\leq K} |U^{k;K}_{ij;n} - U^{k;4K}_{ij;2n}|, \tag{4.5.30}$$

where $U^{k;K}_{ij;n}$ is the numerical solution and $U^{k;4K}_{ij;2n}$ is also a numerical solution but on a finer mesh. Note that we have multiplied n by 2 and K by 4 to balance the error. Since the numerical method constructed here converges at a second order rate in space and a first order in time. Furthermore, we compute the rate of convergence with the formula

$$r^{\varepsilon}_{l} = \log_2 \left(E^{\varepsilon}_{n,\tau}/E^{\varepsilon}_{2n,\tau/4}\right), \quad l = 1, 2, 3, ... \tag{4.5.31}$$

Example 4.5.1. *[4] Consider the problem*

$$u_t - \Delta u + (2 + xy)u = \left(\max\{0, \cos\pi((x - 0.5)^2 + (y - 0.5)^2) - \exp(-t)\}\right)^2, \tag{4.5.32}$$

$$(x, y, t) \in \Omega, \times(0, 1], \quad u(x, y, 0) = 0, \quad x, y \in [0, 1], \quad g(x, y, t) = 0, \; t \in (0, 1]. \tag{4.5.33}$$

Table 4.1: Maximum pointwise error for Example 4.5.1 using the FMFDML

ε	$n = 4$	8	16	32
	$K = 4$	16	64	256
10^{0}	$4.37E-04$	$8.72E-05$	$2.14E-05$	$5.39E-06$
10^{-1}	$3.39E-03$	$9.53E-04$	$2.50E-04$	$6.31E-05$
10^{-2}	$5.98E-03$	$1.68E-03$	$4.34E-04$	$1.09E-04$
10^{-3}	$6.16E-03$	$1.78E-03$	$4.60E-04$	$2.13E-04$
10^{-4}	$6.28E-03$	$1.78E-03$	$4.64E-04$	$2.13E-04$
10^{-5}	$6.29E-03$	$1.78E-03$	$4.65E-04$	$2.13E-04$
10^{-6}	$6.29E-03$	$1.78E-03$	$4.65E-04$	$2.14E-04$
10^{-7}	$6.29E-03$	$1.78E-03$	$4.65E-04$	$2.14E-04$
10^{-8}	$6.28E-03$	$1.78E-03$	$4.65E-04$	$2.14E-04$
10^{-9}	$6.29E-03$	$1.78E-03$	$4.65E-04$	$2.14E-04$
10^{-10}	$6.29E-03$	$1.78E-03$	$4.65E-04$	$2.14E-04$

Table 4.2: Maximum pointwise error for Example 4.5.1 using the FOFDML

ε	$n = 4$	8	16	32
	$K = 4$	16	64	256
10^{0}	$5.24E-04$	$1.11E-04$	$2.81E-05$	$7.66E-06$
10^{-1}	$4.12E-03$	$1.10E-03$	$2.84E-04$	$7.89E-05$
10^{-2}	$6.97E-03$	$2.02E-03$	$5.94E-04$	$1.85E-04$
10^{-3}	$6.48E-03$	$1.63E-03$	$3.73E-04$	$5.89E-04$
10^{-4}	$6.53E-03$	$1.80E-03$	$3.55E-04$	$7.71E-04$
10^{-5}	$6.53E-03$	$1.81E-03$	$4.65E-04$	$9.01E-05$
10^{-6}	$6.53E-03$	$1.81E-03$	$4.65E-04$	$1.17E-04$
10^{-7}	$6.53E-03$	$1.81E-03$	$4.65E-04$	$1.17E-04$
10^{-8}	$6.53E-03$	$1.81E-03$	$4.65E-04$	$1.17E-04$
10^{-9}	$6.53E-03$	$1.81E-03$	$4.65E-04$	$1.17E-04$
10^{-10}	$6.97E-03$	$2.02E-03$	$5.94E-04$	$1.17E-04$

Table 4.3: Rate of convergence for Example 4.5.1 using the FMFDML

ε	r_1	r_2	r_3
10^{0}	2.32	2.02	1.99
10^{-1}	1.83	1.93	1.97
10^{-2}	1.83	1.95	1.98
10^{-3}	1.83	1.95	1.98
10^{-4}	1.78	1.96	1.98
10^{-5}	1.81	1.95	1.98
10^{-6}	1.82	1.94	1.98
10^{-7}	1.82	1.94	1.98
10^{-8}	1.82	1.93	1.98
10^{-9}	1.82	1.93	1.98
10^{-10}	1.82	1.93	1.98

Table 4.4: Rate of convergence for Example 4.5.1 using the FOFDML

ε	r_1	r_2	r_3
10^{0}	2.24	1.98	1.88
10^{-1}	1.91	1.95	1.85
10^{-2}	1.79	1.77	1.68
10^{-3}	1.99	2.13	-0.66
10^{-4}	1.86	2.34	-1.12
10^{-5}	1.85	1.96	2.37
10^{-6}	1.85	1.96	1.99
10^{-7}	1.85	1.96	1.99
10^{-8}	1.85	1.96	1.99
10^{-9}	1.85	1.96	1.99
10^{-10}	1.85	1.96	1.99

4.6 Conclusion

We studied two-dimensional reaction-diffusion problems. After reviewing the FMFDML developed in [12], we designed a FOFDML. The FOFDML consists of a spatial discretization via a FOFDM and a spatial discretization using the backward Euler method. Analyses showed that both methods are second order convergent in space (except for a logarithmic factor in the case of the FMFDML) and first order in time. To illustrate the proposed numerical methods in this chapter we performed numerical simulations, where the maximum pointwise error and rate of convergence were computed. These numerical results are in accordance with the theoretical ones.

In the next chapter we study two-dimensional convection-diffusion problems.

Chapter 5

Methods of Lines for Two-Dimensional Convection-Diffusion Problems

In the preceding chapter we employed the FMFDML and the FOFDML to approximate a two-dimensional time-dependent reaction-diffusion problems. Theoretical error bounds as well as computed numerical results of the designed methods showed that the methods were $\varepsilon-$uniform. In this chapter, we study a two-dimensional time-dependent convection-diffusion problem using the same procedure.

5.1 Continuous problem

Consider the time-dependent convection-diffusion problem

$$\mathcal{L}u \equiv \frac{\partial u}{\partial t} - \varepsilon \Delta u + \mathbf{a}(x,y,t)\nabla u + b(x,y,t)u = f(x,y,t), \ (x,y,t) \in Q \equiv \Omega = (0,1)^2 \times (0,T], \tag{5.1.1}$$

subject to the boundary and initial conditions

$$u(x,y,t) = g(x,y,t), \ (x,y,t) \in \partial\Omega \times [0,T], u(x,y,0) = \varphi(x,y), \ (x,y) \in \bar{\Omega}. \tag{5.1.2}$$

Where ε is a small positive parameter, f and the coefficient functions are sufficiently smooth. We assume $\mathbf{a}(x,y,t) = (a_1(x,y,t), a_2(x,y,t))$, $a_1(x,y,t) \geq \alpha_1 > 0$, $a_2(x,y,t) \geq$

$\alpha_2 > 0$, $b(x,y,t) \geq \beta > 0$, $\forall\ (x,y,t) \in Q$, so that the exact solution satisfies $u(x,y,t) \in \mathcal{C}^{4,2}(Q)$. Also, we impose the compatibility conditions

$$\begin{aligned}
g(x,y,0) &= \varphi(x,y), \text{ in } \partial\Omega, \\
\frac{\partial g}{\partial t}(x,y,0) &= \varepsilon \bigtriangleup \varphi - \mathbf{a}(x,y,0)\nabla\varphi - b(x,y,0)\varphi + f(x,y,0), \text{ in } \partial\Omega, \\
\frac{\partial^2 g}{\partial t^2}(x,y,0) &= (-\varepsilon \bigtriangleup + \mathbf{a}(x,y,0)\nabla + b(x,y,0))^2\varphi + \frac{\partial}{\partial t} f(x,y,0) \\
&\quad - (-\varepsilon \bigtriangleup + \mathbf{a}(x,y,0)\nabla + b(x,y,0))\, f(x,y,0), \text{ in } \partial\Omega, \\
f(x,y,0) &= \frac{\partial}{\partial t} + (-\varepsilon \bigtriangleup + \mathbf{a}(x,y,0)\nabla + b(x,y,0))\, g(x,y,t), \text{ in } (0,1)\times(0,1)\times(0,T].
\end{aligned}$$

The smoothness of the data and the compatibility conditions ensure an outflow boundary layers in the solution of problem (5.1.1)-(5.1.2) these layers are exponential. Figure 5.1 below shows the boundary layers in the solution.

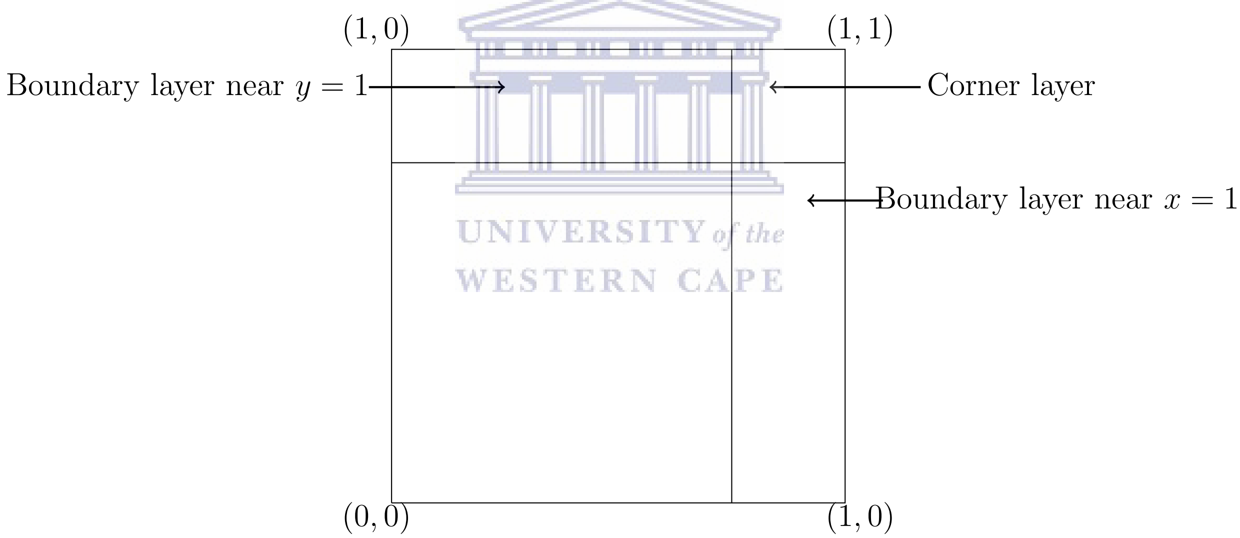

Figure 5.1: Outflow boundary layers in convection-diffusion probem

Thus the differential operator $\mathcal{L}$ satisfies the maximum principle as well as the uniform stability estimate in Lemmas 4.1.1 and 4.1.2 of Chapter 4. Under the hypothesis of these two Lemmas the exact solution and its derivatives satisfy the bound, [42, 12]

$$\left|\frac{\partial^{i+j+k}u(x,y,t)}{\partial x^i \partial y^j \partial t^k}\right| \leq C\left(1 + \varepsilon^{-(i+j)}\left(\exp\left(\frac{-\alpha_1(1-x)}{\varepsilon}\right)\exp\left(\frac{-\alpha_2(1-y)}{\varepsilon}\right)\right)\right). \quad (5.1.3)$$

This bound can further be written as the sum [12]

$$u = v + \sum_{p=1}^{3} w_p, \tag{5.1.4}$$

where v represents the regular component, w_1 and w_2 are the two edge layer functions around the neighbourhood of $x = 1,\ y = 1$ and w_3 is the corner layer function around $(1, 1)$ respectively.

Each term satisfies the respective bounds

$$\frac{\partial^{i+j+k} v(x, y, t)}{\partial x^i \partial y^j \partial t^k} \leq C \tag{5.1.5}$$

$$\frac{\partial^{i+j+k} w_1(x, y, t)}{\partial x^i \partial y^j \partial t^k} \leq C\varepsilon^{-i} \exp\left(-\alpha_1(1-x)/\varepsilon\right) \tag{5.1.6}$$

$$\frac{\partial^{i+j+k} w_2(x, y, t)}{\partial x^i \partial y^j \partial t^k} \leq C\varepsilon^{-j} \exp\left(-\alpha_2(1-y)/\varepsilon\right) \tag{5.1.7}$$

$$\frac{\partial^{i+j+k} w_3(x, y, t)}{\partial x^i \partial y^j \partial t^k} \leq C\varepsilon^{-(i+j)} \min\left\{\exp\left(-\alpha_1(1-x)/\varepsilon\right), \exp\left(-\alpha_2(1-y)/\varepsilon\right)\right\} \tag{5.1.8}$$

where $i + j + 2k \leq 4$. The outline of this chapter is as follows. Section 5.2 describes the spatial discretization with the FMFDM and the error analysis. In Section 5.3, we discretize in space with the FOFDM and provide the error analysis. Section 5.4 integrates the semi-discrete systems of IVPs. Numerical simulations are carried out in Section 5.5 and in Section 5.6, we give a summary of the chapter.

5.2 Spatial discretization with the FMFDM

Below we investigate the FMFDM derived by Clavero and Jorge in [12]. As indicated earlier problem (5.1.1)-(5.1.2) is known to be characterised with an outflow boundary layer of width $\mathcal{O}\varepsilon \ln(1/\varepsilon)$ around the sides $x = 1$ and $y = 1$. Therefore in each spatial direction we design a piecewise uniform mesh to resolve the layers. We use the transition parameters σ_x and σ_y, which are given respectively by

$$\sigma_x = \min\left\{\frac{1}{2}, \sigma_0\frac{\varepsilon}{\alpha_1}\ln n\right\} \quad \text{and } \sigma_y = \min\left\{\frac{1}{2}, \sigma_0\frac{\varepsilon}{\alpha_2}\ln m\right\}.$$

To ensure that there are always some mesh points in every region of the rectangular domain, $n,\ m \geq 4$. In the x direction, the domain $[0,1]$ is sub-divided into $[0, 1-\sigma_x]$ and $[1-\sigma_x, 1]$. With each sub-domain having $n/2+1$ mesh points. This gives the mesh points $\mathbb{I}_{x,n} = \{0 = x_0 < ... < x_n = 1\}$, with the step size

$$\Delta x_i = x_i - x_{i-1} := \begin{cases} 2(1-\sigma_x)n^{-1}, & i = 1, 2, ..., n/2 \\ 2\sigma_x n^{-1}, & i = n/2+1, ..., n. \end{cases}$$

The mesh in the y direction can be obtain in a similar manner. Figure 5.2 illustrates the mesh on the domain $\Omega^{n,m}$.

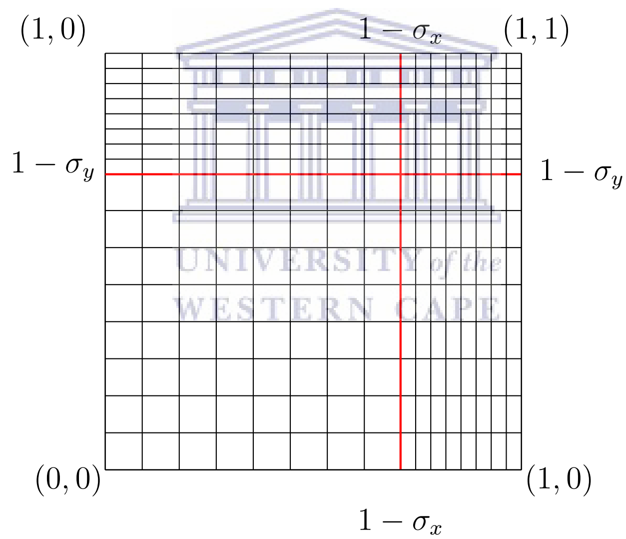

Figure 5.2: Shishkin mesh for $n = 16$ for an elliptic convection-diffusion probem

5.2.1 The FMFDM

Using the FMFDM with the above features and the notations in Chapter 4, we discretize the spatial derivatives in problem (5.1.1)-(5.1.2) as follows:

$$\mathcal{L}^{n,m}U_{ij} \equiv \frac{dU_{ij}(t)}{dt} - \frac{2\varepsilon}{\Delta x_i + \Delta x_{i+1}}\left(\frac{U_{i+1,j}(t) - U_{ij}(t)}{\Delta x_{i+1}} - \frac{U_{ij}(t) - U_{i-1,j}(t)}{\Delta x_i}\right)$$

$$
\begin{aligned}
&-\frac{2\varepsilon}{\Delta y_j+\Delta y_{j+1}}\left(\frac{U_{i,j+1}(t)-U_{ij}(t)}{\Delta y_{j+1}}-\frac{U_{ij}(t)-U_{i,j-1}(t)}{\Delta y_j}\right)+a_{1ij}(t)\\
&\times\left(\frac{U_{ij}(t)-U_{i-1,j}(t)}{\Delta x_i}\right)+a_{2ij}(t)\left(\frac{U_{ij}(t)-U_{i,j-1}(t)}{\Delta y_j}\right)+b_{ij}(t)U_{ij}(t)\\
&=f_{ij}(t),\quad i=1(1)n-1,\ \ j=1(1)m-1, \qquad (5.2.9)
\end{aligned}
$$

with the boundary and the initial conditions

$$
U_{ij}(t)=g_{ij}(t),\ (x_i,y_j,T)\in\partial\Omega^{n,m}\times[0,T],\ \ U_{ij}(0)=\varphi_{ij},\ \ (x_i,y_j)\in\bar{\Omega}^{n,m}. \quad (5.2.10)
$$

We represent the scheme (5.2.9)-(5.2.10) in the matrix notation

$$
U'(t)+A(t)U(t)=F(t), \qquad (5.2.11)
$$

where the coefficient matrix $A(t)$ is a pentadiagonal matrix and satisfies $A(t)\in\mathbb{R}^{(n-1)^2}\times\mathbb{R}^{(m-1)^2}$, and the vectors $U(t)$ and $F(t)$ are in $\mathbb{R}^{(n-1)(m-1)}$. The entries of $A(t)$ and $F(t)$ are given by

$$
\begin{aligned}
A_{pp}(t) &= \frac{2\varepsilon}{\Delta x_i+\Delta x_{i+1}}\left(\frac{1}{\Delta x_{i+1}}+\frac{1}{\Delta x_i}\right)+\frac{2\varepsilon}{\Delta y_j+\Delta y_{j+1}}\left(\frac{1}{\Delta y_{j+1}}+\frac{1}{\Delta y_j}\right)+\frac{a_{1ij}(t)}{\Delta x_i}\\
&\quad+\frac{a_{2ij}(t)}{\Delta y_j}+b_{ij}(t), \qquad i=1(1)n-1,\quad j=1(1)m-1,\\
A_{p,p-1}(t) &= -\frac{2\varepsilon}{\Delta y_j+\Delta y_{j+1}}\left(\frac{1}{\Delta y_j}\right)-\frac{a_{2ij}(t)}{\Delta y_j}, \qquad i=1(1)n-1,\quad j=2(1)m-1,\\
A_{p,p+1}(t) &= -\frac{2\varepsilon}{\Delta y_j+\Delta y_{j+1}}\left(\frac{1}{\Delta y_{j+1}}\right), \qquad i=1(1)n-1,\quad j=1(1)m-2,\\
A_{p,p-(n-1))}(t) &= -\frac{2\varepsilon}{\Delta x_i+\Delta x_{i+1}}\left(\frac{1}{\Delta x_i}\right)-\frac{a_{1ij}(t)}{\Delta x_i}, \qquad i=2(1)n-1,\quad j=1(1)m-1,\\
A_{p,(n-1)+p}(t) &= -\frac{2\varepsilon}{\Delta x_i+\Delta x_{i+1}}\left(\frac{1}{\Delta x_{i+1}}\right), \qquad i=1(1)n-2,\quad j=1(1)m-1.
\end{aligned}
$$

$$
\begin{aligned}
F_p(t)&=f_p(t)+\left(\frac{2\varepsilon}{\Delta x_1(\Delta x_1+\Delta x_2)}+\frac{a_{111}(t)}{\Delta x_1}\right)u(0,y_1,t)+\left(\frac{2\varepsilon}{\Delta y_1(\Delta y_1+\Delta y_2)}+\frac{a_{211}(t)}{\Delta y_1}\right)\\
&\quad\times u(x_1,0,t),\\
F_p(t) &= f_p(t)+\left(\frac{2\varepsilon}{\Delta x_1(\Delta x_1+\Delta x_2)}+\frac{a_{11j}(t)}{\Delta x_1}\right)u(0,y_j,t), \quad j=2(1)m-2,\\
F_p(t) &= f_p(t)+\left(\frac{2\varepsilon}{\Delta x_1(\Delta x_1+\Delta x_2)}+\frac{a_{11,m-1}(t)}{\Delta x_1}\right)u(0,y_{m-1},t)+\left(\frac{2\varepsilon}{\Delta y_m(\Delta y_{m-1}+\Delta y_m)}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \times u(x_1,1,t),\\
F_p(t) &= f_p(t)+\left(\frac{2\varepsilon}{\Delta y_1(\Delta y_1+\Delta y_2)}+\frac{a_{2i1}(t)}{\Delta y_1}\right)u(x_i,0,t), \quad i=2(1)n-2,\\
F_p(t) &= f_p(t), \qquad\qquad i=2(1)n-2,\ \ j=2(1)m-2,\\
F_p(t) &= f_p(t)+\left(\frac{2\varepsilon}{\Delta y_m(\Delta y_{m-1}+\Delta y_m)}\right)u(x_i,1,t), \qquad i=2(1)n-2,\\
F_p(t) &= f_p(t)+\left(\frac{2\varepsilon}{\Delta x_n(\Delta x_{n-1}+x_n)}\right)u(1,y_1,t)+\left(\frac{2\varepsilon}{\Delta y_1(\Delta y_1+\Delta y_2)}+\frac{a_{211}(t)}{\Delta y_1}\right)\\
& u(x_{n-1},0,t),\\
F_p(t) &= f_p(t)+\left(\frac{2\varepsilon}{\Delta x_n(\Delta x_{n-1}+\Delta x_n)}\right)u(1,y_j,t), \qquad j=2(1)m-2,\\
F_p(t) &= f_p(t)+\left(\frac{2\varepsilon}{\Delta x_n(\Delta x_{n-1}+\Delta x_n)}\right)u(1,y_{m-1},t)+\left(\frac{2\varepsilon}{\Delta y_m(\Delta y_{m-1}+\Delta y_m)}\right)\\
& u(x_{n-1},1,t).
\end{aligned}
$$

Next we analyse the FMFDM for convergence.

5.2.2 Error analysis

Here we give the theoretical error analysis of the scheme (5.2.9)-(5.2.10). Before we embark on the error analysis, we first present two lemmas which will be used in the analysis.

Lemma 5.2.1. *(Semi-discrete maximum principle) Let $\mathcal{L}^{m,n}$ be as defined in (5.2.9). We say $\mathcal{L}^{n,m}$ satisfies a semi-discrete maximum principle if $\Psi_{0j}(t)\geq 0,\ \ \Psi_{i0}(t)\geq 0,\ \ \Psi_{nj}(t)\geq 0,\ \ \Psi_{im}(t)\geq 0,\ \ \forall\ 0\leq i\leq n,\ \ 0\leq j\leq m,\ \ 0\leq t\leq T$ and $\mathcal{L}^{n,m}\Psi_{ij}(t)\geq 0,\ \ 0<i<n,\ \ 0<j<m,\ \ 0\leq t\leq T$.*

Proof. Let (l,s) be indices such that

$$\Psi_{ls}(t)=\min_{(x_i,y_j,t)\in\bar{\Omega}^{n,m}\times[0,T]}\Psi_{ij}(t)\ \text{ and }\ \Psi_{ls}(t)<0.$$

Then the indices (l,s) lie in the interval $0<l<n,\ \ 0<s<m$. We see that $(\Psi_{ls}(t))_t=0,\ \Psi_{l+1,s}(t)>\Psi_{ls}(t),\ \ \Psi_{ls}(t)>\Psi_{l-1,s}(t),\ \ \Psi_{l,s+1}(t)>\Psi_{ls}(t),\ \ \Psi_{ls}(t)>\Psi_{l,s-1}(t)$, thus $\mathcal{L}^{n,m}\Psi_{ls}(t)<0$, which is a contradiction. □

Lemma 5.2.2. *The solution $u_{ij}(t)$ of the semi-discrete problem (5.2.9)-(5.2.10), is such that*

$$||u_{ij}(t)|| \ \leq\ \beta^{-1}\max_{(x_i,y_j,t)\in\bar{\Omega}^{n,m}\times[0,T]}||\mathcal{L}^{n,m}u_{ij}(t)||+\max_{(x_i,y_j,t)\in\bar{\Omega}^{n,m}\times[0,T]}(||\varphi_{ij}||,g_{ij}(t)).$$

Proof. We let $p = \beta^{-1} \max_{(x_i,y_j,t)\in\bar{\Omega}^{n,m}\times[0,T]} ||\mathcal{L}^{n,m}u_{ij}(t)|| + \max_{(x_i,y_j,t)\in\bar{\Omega}^{n,m}\times[0,T]}(||\varphi_{ij}||, g_{ij}(t))$, and define the function $\psi_{ij}^{\pm}(t)$ by

$$\psi_{ij}^{\pm}(t) = p \pm u_{ij}(t).$$

When we consider the boundary conditions, we obtain

$$\begin{aligned}
\psi_{0j}^{\pm}(t) &= p \pm u_{0j}(t) = p \pm g_{0j}(t) \geq 0,\\
\psi_{nj}^{\pm}(t) &= p \pm u_{nj}(t) = p \pm g_{nj}(t) \geq 0,\\
\psi_{i0}^{\pm}(t) &= p \pm u_{i0}(t) = p \pm g_{i0}(t) \geq 0,\\
\psi_{im}^{\pm}(t) &= p \pm u_{im}(t) = p \pm g_{im}(t) \geq 0,
\end{aligned}$$

and for the domain $\Omega^{n,m} \times [0,T]$, we have

$$\begin{aligned}
\mathcal{L}^{n,m}\psi_{ij}^{\pm}(t) &= (p \pm u_{ij}(t))' - \varepsilon\left[\frac{2}{\Delta x_i + \Delta x_{i+1}}\left(\frac{p \pm u_{i+1,j}(t) - p \pm u_{ij}(t)}{\Delta x_{i+1}} - \frac{p \pm u_{ij}(t) - p \pm u_{i-1,j}(t)}{\Delta x_i}\right)\right.\\
&\quad \left. + \frac{2}{\Delta y_j + \Delta y_{j+1}}\left(\frac{p \pm u_{i,j+1}(t) - p \pm u_{ij}(t)}{\Delta y_{j+1}} - \frac{p \pm u_{ij}(t) - p \pm u_{i,j-1}(t)}{\Delta y_j}\right)\right]\\
&\quad + a_{1ij}(t)\left(\frac{p \pm u_{ij}(t) - p \pm u_{i-1,j}(t)}{\Delta x_i}\right) + a_{2ij}(t)\left(\frac{p \pm u_{ij}(t) - p \pm u_{i,j-1}(t)}{\Delta y_j}\right)\\
&\quad + b_{ij}(t)(p \pm u_{ij}(t))\\
&= b_{ij}(t)p \pm \mathcal{L}^{n,m}u_{ij}(t)\\
&= b_{ij}(t)p \pm f_{ij}(t)\\
&= b_{ij}(t)\left[\beta^{-1}\max_{(x_i,y_j,t)\in\Omega^{n,m}\times[0,T]}||\mathcal{L}^{n,m}u_{ij}(t)|| + \max_{(x_i,y_j,t)\in\Omega^{n,m}\times[0,T]}(||\varphi_{ij}||, g_{ij}(t))\right] \pm f_{ij}(t)\\
&\geq 0, \text{ since } b_{ij}(t) \geq \beta.
\end{aligned}$$

It follow from the semi-discrete maximum principle 5.2.1 that $\psi_{ij}(t) \geq 0, \; \forall\, 0 \leq i \leq n, \; 0 \leq j \leq m, \; 0 \leq t \leq T$, which ends the proof. □

For simplicity we set $n = m$ through out the analysis. To be able to carry out a rigorous analysis of the FMFDM, we split the numerical solution into

$$U_{ij}(t) = V_{ij}(t) + \sum_{p=1}^{3} W_{p_{ij}}(t), \quad \forall\,(x_i, y_j) \in \Omega^n, \; t \in (0, T],$$

where $V_{ij}(t)$ is a solution to

$$\mathcal{L}^n V_{ij}(t) = f_{ij}(t), \; V_{ij}(0) = v_{ij}(0), \tag{5.2.12}$$

and $W_{p_{ij}}(t)$ also satisfies

$$\mathcal{L}^n W_{p_{ij}}(t) = g_{p_{ij}}(t) + \frac{\partial g_{ij}(t)}{\partial t}, \;\; W_{p_{ij}}(0) = w_{p_{ij}}, \;\; p = 1, 2, 3. \tag{5.2.13}$$

Where $g_{1_{ij}}(t)$, $g_{2_{ij}}(t)$ are some evaluated boundary conditions around the sides $x = 1$ and $y = 1$ and $g_{3_{ij}}(t)$ contains evaluated boundary conditions around the corner $(1, 1)$. Having obtained this decomposition, we estimate the error in each component separately.

Error of the regular component

We write the error in this component as

$$\begin{aligned}
\mathcal{L}^n(V_{ij}(t) - v_{ij}(t)) &= f_{ij}(t) - \mathcal{L}^n v_{ij}(t) \\
&= (\mathcal{L} - \mathcal{L}^n) v_{ij}(t) \\
&= -\varepsilon \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - (\delta_x^2 + \delta_y^2) \right) v_{ij}(t) + a_{1ij}(t) \left(\frac{\partial}{\partial x} - D_x^- \right) v_{ij}(t) \\
&\quad + a_{2ij}(t) \left(\frac{\partial}{\partial y} - D_y^- \right) v_{ij}(t).
\end{aligned}$$

Using Taylor series expansions with integral remainder term simplifies the truncation error into

$$\begin{aligned}
||\mathcal{L}^n(V(t) - v(t))(x_i, y_j)|| &\leq -C_1\varepsilon[(x_{i+1} - x_{i-1})||(v_{xxx}(t))_{ij}|| + (y_{j+1} - y_{j-1})||(v_{yyy}(t))_{ij}||] \\
&\quad + C_2 \left[a_{1ij}(t)\Delta x_i ||(v_{xx}(t))_{ij}|| + a_{2ij}(t)\Delta y_j ||(v_{yy}(t))_{ij}|| \right] \\
&\leq C\varepsilon n^{-1} + Cn^{-1} \\
&\leq Cn^{-1},
\end{aligned}$$

where we have used appropriate bounds of the derivatives. Now we follow [12] to define the barrier function $\psi_{v_{ij}}(t) = Cn^{-1}(x_i + y_j)$, which satisfies $\psi_{v_{ij}}(t) \geq 0, \; \forall \; (x_i, y_j, t) \in \bar{\Omega}^n \times [0, T]$. Thus it follows that

$$||(V_{ij}(t) - v_{ij}(t))|| \leq Cn^{-1}.$$

Error around the edges of the domain

To bound the boundary layer term $W_{1_{ij}}(t)$, we consider the two sub-domains $[0, 1 - \sigma_x] \times [0, 1] \times [0, T]$ and $[1 - \sigma_x, 1] \times [0, 1] \times [0, T]$ in the x direction. That is the layer region and the non-layer region in the x direction. In the sub-domain $[0, 1-\sigma_x] \times [0,1] \times [0,T]$, the mesh is very coarse, thus to bound the error we follow [26] to define the discrete barrier function

$$Z_{1ij}(t) = \prod_{s=1}^{i}\left(1 + \frac{\alpha_1 \Delta x_s}{\varepsilon}\right), \text{ on } \Omega^n, \ t \in (0,T].$$

Similar to the one-dimensional case $Z_{1ij}(t)$ satisfies the inequalities

$$\mathcal{L}^n Z_{1ij}(t) \geq \frac{C_1}{\varepsilon + \alpha_1 \Delta x_i} Z_{1ij}(t), \text{ on } \Omega^n, \quad (5.2.14)$$

$$\exp(-\alpha_1(1-x_i)/\varepsilon) = \prod_{s=i+1}^{n} \exp(-\alpha_1 \Delta x_s/\varepsilon) \leq \prod_{s=i+1}^{n}\left(1+\frac{\alpha_1\Delta x_s}{\varepsilon}\right)^{-1}$$

$$= \left[\prod_{s=1}^{n}\left(1+\frac{\alpha_1\Delta x_s}{\varepsilon}\right)^{-1}\right] Z_{1ij}(t) \quad (5.2.15)$$

Now we let $\psi_{1ij}(t) = C_1 \left[\prod_{s=1}^{n}\left(1+\frac{\alpha\Delta x_s}{\varepsilon}\right)^{-1}\right] Z_{1ij}(t)$, and show that $\psi_{1ij}(t)$ is a barrier function for $W_{1ij}(t)$ when C_1 is chosen to be sufficiently large. At the boundaries we have

$$\begin{aligned} W_{10j}(t) &= w(0,y_j,t) \leq \exp(-\alpha_1/\varepsilon) = C\prod_{s=1}^{n}\exp(-\alpha_1\Delta x_s/\varepsilon) \leq C\prod_{s=1}^{n}\left(1+\frac{\alpha_1\Delta x_s}{\varepsilon}\right)^{-1} \\ W_{1nj}(t) &= w(1,y_j,t) \leq C, \end{aligned}$$

for $W_{1ij}(t)$ and

$$\begin{aligned} \psi_{10j}(t) &= C_1\prod_{s=1}^{n}\left(1+\frac{\alpha_1\Delta x_s}{\varepsilon}\right)^{-1} \geq W_{10j}(t) \\ \psi_{1nj}(t) &= C_1\prod_{s=1}^{n}\left(1+\frac{\alpha_1\Delta x_s}{\varepsilon}\right)^{-1} Z_{1nj}(t) \geq W_{1nj}(t), \end{aligned}$$

also holds when C_1 is chosen to be sufficiently large. From the inequality (5.2.14)

$$\mathcal{L}^n\psi_{1ij}(t) \geq \frac{C_1}{\varepsilon+\alpha_1\Delta x_i}\left[\prod_{s=1}^{n}\left(1+\frac{\alpha_1\Delta x_s}{\varepsilon}\right)^{-1}\right] Z_{1ij}(t) \geq 0\mathcal{L}^n W_{ij}(t) \qquad (5.2.16)$$

holds. Thus $\psi_{1ij}(t)$ is a barrier function for $W_{1ij}(t)\ \forall\,(x_i,y_j) \in \Omega^n, \ t \in (0,T]$. Employing the bounds 5.1.6 we have the estimate

$$\begin{aligned} ||W_{1ij}(t) - w_{1ij}(t)|| \leq ||w_{1ij}(t)|| + ||W_{1ij}(t)|| &\leq C\exp(-\alpha(1-x_i)/\varepsilon) + \psi_{1ij}(t) \\ &\leq C\psi_{1ij}(t) \ \ \forall \ (x_i,y_j,t) \in \Omega^n \times (0,T]. \end{aligned}$$

Now for all $0 \le i \le n/2$, we show that $\psi_{1ij}(t) \le Cn^{-1}$. To do this it is suffices to show that $\psi_{1n/2j}(t) \le Cn^{-1}$ since $\psi_{1ij}(t) \le \psi_{1i+1,j}(t)$ for $0 \le i \le n/2,\ 0 \le j \le n,\ 0 \le t \le T$. Thus for all $0 \le i \le n/2,\ 0 \le j \le n,\ 0 \le t \le T$, we have

$$\begin{aligned}
Z_{1ij}(t) \le Z_{1n/2,j}(t) &= \prod_{s=1+n/2}^{n}\left(1+\frac{\alpha_1 \Delta x_s}{\varepsilon}\right)^{-1} \\
&= C\exp(-\alpha_1(1-x_{n/2})/\varepsilon) \\
&= C\exp(-\alpha-1(1-(1-\sigma))/\varepsilon) \\
&= C\exp(-\alpha_1(\sigma))/\varepsilon) \\
&= C\exp(-\alpha_1(\varepsilon/\alpha_1 \ln n)/\varepsilon) \\
&\le Cn^{-1}.
\end{aligned}$$

In the other half of the domain $n/2 < i < n$, $0 \le j \le n$, the mesh is very fine therefore we bound the error with consistency and the barrier function argument. Thus we have the truncation error in this region as

$$\begin{aligned}
\mathcal{L}^n(W_{1ij}(t) - w_{1ij}(t)) &= -\varepsilon\left[\left(\frac{\partial^2}{\partial x^2} - \delta_x^2\right) + \left(\frac{\partial^2}{\partial y^2} - \delta_y^2\right)\right] w_{1ij}(t) \\
&\quad + \left[a_{1ij}(t)\left(\frac{\partial}{\partial x} - D_x^-\right) + a_{2ij}(t)\left(\frac{\partial}{\partial y} - D_y^-\right)\right] w_{1ij}(t) \\
&\le -C\varepsilon[(x_{i+1} - x_{i-1})(w_{1xxx}(t))_{ij} + (y_{j+1} - y_{j-1})(w_{1yyy}(t))_{ij}] \\
&\quad + a_{1ij}(t)\Delta x_i[w_{1xx}(t))_{ij}] + a_{2ij}(t)\Delta y_j[w_{1yy}(t))_{ij}] \\
&\le C\sigma n^{-1}\varepsilon^{-2}\exp(-\alpha_1(1-x)/\varepsilon) + C\varepsilon n^{-1} + Cn^{-1} \\
&\le Cn^{-1}\ln n\varepsilon^{-1}\exp(-\alpha_1(1-x)/\varepsilon) + C\varepsilon n^{-1} + Cn^{-1} \\
&\le Cn^{-1}\ln n\varepsilon^{-1}\left[\prod_{s=i+1}^{n}\left(1+\frac{\alpha_1 \Delta x_s}{\varepsilon}\right)^{-1}\right],
\end{aligned}$$

where the bounds 5.1.6 and $\Delta x = 2\sigma_x n^{-1}$ have been used. Now we define the barrier function

$$\psi_{1ij}(t) = Cn^{-1}\ln n\prod_{s=i+1}^{n}\left(1+\frac{\alpha_1 \Delta x_s}{\varepsilon}\right)^{-1}. \tag{5.2.17}$$

Which satisfies the inequalities

$$\psi_{1n/2j}(t) = Cn^{-1}\ln n\left(1+\frac{\alpha_1\sigma_x}{n\varepsilon}\right)^{-1} \le Cn^{-1}\ln n \ge 0,$$

$$
\begin{aligned}
\psi_{1nj}(t) &= Cn^{-1}\ln n\left(1+\frac{\alpha_1\sigma_x}{n\varepsilon}\right)^{-1} \le Cn^{-1}\ln n \ge 0,\ \forall\, n/2 \le i \le n \\
& \quad 0 \le j \le n,\ t, \in\ [0,T], \\
\mathcal{L}^n\psi_{1ij}(t) &= Cn^{-1}\ln n\left(1+\frac{\alpha_1\sigma_x}{n\varepsilon}\right)^{-1} \le Cn^{-1}\ln n \ge 0,\ \ \forall\, n/2 < i < n, \\
& \quad 0 < j < n\,,\ t \in [0,T].
\end{aligned}
$$

Therefore $\psi_{1ij}(t) \le Cn^{-1}\ln n \ge 0,\ \ \forall\, n/2 \le i \le n,\ 0 \le j \le n,\ t \in (0,T]$. It follows that

$$
||W_{1ij}(t) - w_{1ij}(t)|| \le\ \ Cn^{-1}\ln n,\ \ n/2 < i < n,\ 0 \le j \le n,\ t \in [0,T].
$$

Combining the results in the two sub-domains gives the bound

$$
||W_{1ij}(t) - w_{1ij}(t)|| \le Cn^{-1}\ln n,\ \ \forall\ 0 \le i,\, j \le n\ \ t \in (0,T]. \tag{5.2.18}
$$

Analogous bound holds for the singular component in the y direction.

$$
||W_{2ij}(t) - w_{2ij}(t)|| \le Cn^{-1}\ln n,\ \ \forall\ 0 \le i,\, j \le n,\, t \in [0,T]. \tag{5.2.19}
$$

Error around the corners of the domain

Now to bound the corner layer function $W_{3ij}(t)$ we consider the layer region and the non-layer region and follow the same procedure as the case of the bound $W_{1ij}(t)$. In the non-layer region, the mesh is coarse thus we follow [26] to define the discrete barrier function

$$
Z_{3ij}(t) = \prod_{s=1}^{i}\left(1+\frac{\alpha_1\Delta x_s}{\varepsilon}\right)\prod_{s=1}^{j}\left(1+\frac{\alpha_2\Delta x_s}{\varepsilon}\right),\ \text{ on } \Omega^n,
$$

which is also the discrete equivalent of the boundary layer term

$$
\exp\left(-\alpha_1(1-x)/\varepsilon\right)\exp\left(-\alpha_2(1-y)/\varepsilon\right).
$$

Similarly, it satisfies the inequalities

$$
\mathcal{L}^n Z_{3ij}(t) \ge C Z_{3ij}(t)\left(\frac{1}{\varepsilon+\alpha_1\Delta x_i} + \frac{1}{\varepsilon+\alpha_2\Delta y_j}\right)
$$

and

$$
\exp\left(-\alpha_1(1-x)/\varepsilon\right)\exp\left(-\alpha_2(1-y)/\varepsilon\right) \ \ \le\ \ \prod_{s=i+1}^{n}\left(1+\frac{\alpha_1\Delta x_s}{\varepsilon}\right)^{-1}\prod_{s=j+1}^{n}\left(1+\frac{\alpha_2\Delta y_s}{\varepsilon}\right)^{-1},
$$

holds.

Proof.

$$\begin{aligned}
D_x^+ Z_{3ij}(t) &= \frac{(1+\alpha_1\Delta x_{i+1}/\varepsilon)\left(\prod_{s=1}^{i}(1+\alpha_1\Delta x_s/\varepsilon)\prod_{s=1}^{j}(1+\alpha_2\Delta y_s/\varepsilon)\right)}{\Delta x_{i+1}} \\
&\quad - \frac{\left(\prod_{s=1}^{i}(1+\alpha_1\Delta x_s/\varepsilon)\prod_{s=1}^{j}(1+\alpha_2\Delta y_s/\varepsilon)\right)}{\Delta x_{i+1}} \\
&= \frac{\left(\prod_{s=1}^{i}(1+\alpha_1\Delta x_s/\varepsilon)\prod_{s=1}^{j}(1+\alpha_2\Delta y_s/\varepsilon)\right)[(1+\alpha_1\Delta x_{i+1}/\varepsilon)-1]}{\Delta x_{i+1}} \\
&= \frac{\alpha_1}{\varepsilon} Z_{3ij}(t), \qquad (5.2.20)
\end{aligned}$$

$$\begin{aligned}
D_x^- Z_{3ij}(t) &= \frac{(1+\alpha_1\Delta x_{i}/\varepsilon)\left(\prod_{s=1}^{i-1}(1+\alpha_1\Delta x_s/\varepsilon)\prod_{s=1}^{j}(1+\alpha_2\Delta y_s/\varepsilon)\right)}{\Delta x_{i}} \\
&\quad - \frac{\left(\prod_{s=1}^{i-1}(1+\alpha_1\Delta x_s/\varepsilon)\prod_{s=1}^{j}(1+\alpha_2\Delta y_s/\varepsilon)\right)}{\Delta x_{i}} \\
&= \frac{\left(\prod_{s=1}^{i}(1+\alpha_1\Delta x_s/\varepsilon)\prod_{s=1}^{j}(1+\alpha_2\Delta y_s/\varepsilon)\right)[(1+\alpha_1\Delta x_{i}/\varepsilon)-1]}{\Delta x_{i}} \\
&= \frac{\alpha_1}{\varepsilon+\alpha_1\Delta x_i} Z_{3ij}(t), \qquad (5.2.21)
\end{aligned}$$

$$\begin{aligned}
D_y^+ Z_{3ij}(t) &= \frac{(1+\alpha_2\Delta y_{j+1}/\varepsilon)\left(\prod_{s=1}^{i}(1+\alpha_1\Delta x_s/\varepsilon)\prod_{s=1}^{j}(1+\alpha_2\Delta y_s/\varepsilon)\right)}{\Delta y_{j+1}} \\
&\quad - \frac{\left(\prod_{s=1}^{i}(1+\alpha_1\Delta x_s/\varepsilon)\prod_{s=1}^{j}(1+\alpha_2\Delta y_s/\varepsilon)\right)}{\Delta y_{j+1}} \\
&= \frac{\left(\prod_{s=1}^{i}(1+\alpha_1\Delta x_s/\varepsilon)\prod_{s=1}^{j}(1+\alpha_2\Delta y_s/\varepsilon)\right)[(1+\alpha_2\Delta y_{j+1}/\varepsilon)-1]}{\Delta y_{j+1}} \\
&= \frac{\alpha_2}{\varepsilon} Z_{3ij}(t), \qquad (5.2.22)
\end{aligned}$$

$$
\begin{aligned}
D_y^- Z_{3ij}(t) &= \frac{(1+\alpha_1 \Delta y_j/\varepsilon)\left(\prod_{s=1}^{i-1}(1+\alpha_1\Delta x_s/\varepsilon)\prod_{s=1}^{j}(1+\alpha_2\Delta y_s/\varepsilon)\right)}{\Delta y_j} \\
&\quad - \frac{\left(\prod_{s=1}^{i-1}(1+\alpha_1\Delta x_s/\varepsilon)\prod_{s=1}^{j}(1+\alpha_2\Delta y_s/\varepsilon)\right)}{\Delta y_j} \\
&= \frac{\left(\prod_{s=1}^{i}(1+\alpha_1\Delta x_s/\varepsilon)\prod_{s=1}^{j}(1+\alpha_2\Delta y_s/\varepsilon)\right)[(1+\alpha_2\Delta y_j/\varepsilon)-1]}{\Delta y_j} \\
&= \frac{\alpha_2}{\varepsilon+\alpha_2\Delta y_j} Z_{3ij}(t), \qquad (5.2.23)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}^n Z_{3ij}(t) &= (Z_{3ij}(t))_t - \varepsilon\left[\frac{2}{\Delta x_{i+1}+\Delta x_i}\left(D_x^+ - D_x^-\right) + \frac{2}{\Delta y_{j+1}+\Delta y_j}\left(D_y^+ - D_y^-\right)\right] \\
&\quad + a_{1ij}(t)D_x^- + a_{2ij}(t)D_y^- + b_{ij}(t)Z_{3ij}(t) \\
&= -\varepsilon\left[\frac{2}{\Delta x_{i+1}+\Delta x_i}\left(\frac{\alpha_1}{\varepsilon} - \frac{\alpha_1}{\varepsilon+\alpha_1\Delta x_i}\right) + \frac{2}{\Delta y_{j+1}+\Delta y_j}\left(\frac{\alpha_2}{\varepsilon} - \frac{\alpha_2}{\varepsilon+\alpha_2\Delta y_j}\right)\right. \\
&\quad \left. + a_{1ij}(t)\left(\frac{\alpha_1}{\varepsilon+\alpha_1\Delta x_i}\right) + a_{2ij}(t)\left(\frac{\alpha_2}{\varepsilon+\alpha_2\Delta y_j}\right) + b_{ij}(t)\right] Z_{3ij}(t) \\
&= \left[\frac{1}{\varepsilon+\alpha_1\Delta x_i}\left(-\frac{2\alpha_1^2\Delta x_i}{\Delta x_i+\Delta x_{i+1}} + \alpha_1^2\right) + \frac{1}{\varepsilon+\alpha_2\Delta y_j}\left(-\frac{2\alpha_2^2\Delta y_j}{\Delta y_j+\Delta y_{j+1}} + \alpha_2^2\right) + \beta\right] \\
&\quad \times Z_{3ij}(t) \\
&\geq \left[\frac{C}{\varepsilon+\alpha_1\Delta x_i} + \frac{C}{\varepsilon+\alpha_2\Delta y_j} + \beta\right] Z_{3ij}(t) \\
&\geq C Z_{3ij}(t)\left(\frac{1}{\varepsilon+\alpha_1\Delta x_i} + \frac{1}{\varepsilon+\alpha_2\Delta y_j}\right), \text{ on } \Omega^n, \qquad (5.2.24)
\end{aligned}
$$

which completes the proof. □

Now we set

$$
\begin{aligned}
\psi_{3ij}(t) &= C_1 \prod_{s=i+1}^{n}\left(1+\frac{\alpha_1\Delta x_s}{\varepsilon}\right)^{-1} \prod_{s=j+1}^{n}\left(1+\frac{\alpha_2\Delta y_s}{\varepsilon}\right)^{-1} \\
&= C_1 \prod_{s=1}^{n}\left(1+\frac{\alpha_1\Delta x_s}{\varepsilon}\right)^{-1} \prod_{s=1}^{n}\left(1+\frac{\alpha_2\Delta y_s}{\varepsilon}\right)^{-1} Z_{3ij}(t),
\end{aligned}
$$

and show that $\psi_{3ij}(t)$ is a barrier function of $W_{3ij}(t)$ when C_1 is chosen to be large. At the boundaries we have

$$\begin{aligned}
\psi_{300}(t) &= C_1 \prod_{s=1}^{n}\left(1+\frac{\alpha_1 \Delta x_s}{\varepsilon}\right)^{-1}\prod_{s=1}^{n}\left(1+\frac{\alpha_2 \Delta y_s}{\varepsilon}\right)^{-1} \geq \exp(-\alpha_1/\varepsilon)\exp(-\alpha_2/\varepsilon) = W_{300}(t) \\
\psi_{3nn}(t) &= C_1 \prod_{s=1}^{n}\left(1+\frac{\alpha_1 \Delta x_s}{\varepsilon}\right)^{-1}\prod_{s=1}^{n}\left(1+\frac{\alpha_2 \Delta y_s}{\varepsilon}\right)^{-1} Z_{3nn}(t) \geq C = W_{3nn}(t).
\end{aligned}$$

From equation (5.2.13) we have $\mathcal{L}^n W_{3ij}(t) = g_{3ij}(t) + \frac{\partial g_{ij}(t)}{\partial t}$, and from inequality (5.2.24) $\mathcal{L}^n \psi_{3ij}(t) \geq C Z_{3ij}(t)\left(\frac{1}{\varepsilon+\alpha_1\Delta x_i}+\frac{1}{\varepsilon+\alpha_2\Delta y_j}\right) \geq \mathcal{L}^n W_{3ij}(t)$, also holds. Therefore $\psi_{3ij}(t)$ is a barrier function for $W_{3ij}(t)$, and hence we have $W_{3ij}(t) \leq \psi_{3ij}(t)$. Thus the error gives

$$\begin{aligned}
||W_{3ij}(t) - w_{3ij}(t)|| \leq C\psi_{ij}(t) &= \prod_{s=i+1}^{n}\left(1+\frac{\alpha_1 \Delta x_s}{\varepsilon}\right)^{-1}\prod_{s=j+1}^{n}\left(1+\frac{\alpha_2 \Delta_s}{\varepsilon}\right)^{-1} \\
&= C\min(\exp(-\alpha_1(1-x_{n/2})/\varepsilon)\exp(-\alpha_2(1-y_{n/2})/\varepsilon)) \\
&= C\min(\exp(-\alpha_1(1-(1-\sigma_x))/\varepsilon) \\
&\quad \exp(-\alpha_2-1(1-(1-\sigma_y))/\varepsilon)) \\
&= C\min(\exp(-\alpha_1(\sigma_x))/\varepsilon)C\exp(-\alpha_2(\sigma_y))/\varepsilon)) \\
&= C\min\exp(-\alpha_1(\varepsilon/\alpha_1\ln n)/\varepsilon)\exp(-\alpha_2(\varepsilon/\alpha_2\ln n)/\varepsilon)) \\
&\leq Cn^{-1}.
\end{aligned}$$

The error satisfies

$$||W_{3ij}(t) - w_{3ij}(t)|| \leq Cn^{-1} \;\; \forall\; 0 \leq i+j \leq \frac{3n}{2} \;\; 0 \leq t \leq T.$$

In the other half of the domain we define the truncation error as

$$\begin{aligned}
||\mathcal{L}^n(W_{3ij}(t) - w_{3ij}(t))|| &\leq C\varepsilon(x_{i+1}-x_{1-1})||(w_{3xxx}(t))_{ij}|| + (y_{j+1}-y_{j-1})||(w_{3yyy}(t))_{ij}|| \\
&\leq C\varepsilon[\Delta x_i(w_{3xxx}(t))_{ij} + \Delta y_j(w_{3yyy}(t))_{ij}] \\
&\leq Cn^{-1}\ln n.
\end{aligned}$$

From the results obtained from each component, the error of the FMFDM satisfies

$$||U_{ij}^k - u_{ij}^k|| \leq Cn^{-1}\ln n.$$

Next we perform the spatial discretization with the FOFDM.

5.3 Spatial discretization with the FOFDM

Below we discretize problem (5.1.1)-(5.1.2) in space via the FOFDM on a uniform mesh.

5.3.1 The FOFDM

Following the same partitioning of the space domain in Section 4.3.1 of Chapter 4, we discretize in space as follows:

$$\mathcal{L}^{\Delta x,\Delta y}U_{ij}(t) \equiv \frac{dU_{ij}(t)}{dt} - \varepsilon\left[\frac{U_{i+1,j}(t)-2U_{ij}(t)-U_{i-1,j}(t)}{(\phi_{ij})^2(\Delta x,\varepsilon,t)} + \frac{U_{i,j+1}(t)-2U_{ij}(t)-U_{i,j-1}(t)}{(\phi_{ij})^2(\Delta y,\varepsilon,t)}\right]$$

$$+a_{1ij}(t)\left[\frac{U_{ij}(t)-U_{i-1,j}(t)}{\Delta x}\right] + a_{2ij}(t)\left[\frac{U_{ij}(t)-U_{i,j-1}(t)}{\Delta y}\right]$$

$$+b_{ij}(t)U_{ij}(t) = f_{ij}(t), \quad \text{for} \ \ i=1,2,...,n-1, \ \ j=1,2,...,m-1, \quad (5.3.25)$$

along with the semi-discrete initial and boundary conditions

$$U_{ij}(0)=\varphi_{ij}, \ \ (x_i,y_j)\in\bar{\Omega}^{n,m}, \ \ U_{ij}(t)=g_{ij}(t)\in \ \partial\Omega^{n,m}\times[0,T], \quad \text{respevtively.} \quad (5.3.26)$$

Here the denominator functions $(\phi_{ij})^2$ are given by

$$(\phi_{ij})^2(\Delta x,\varepsilon,t) = \frac{\Delta x\varepsilon}{a_{1ij}(t)}\left(\exp\left(\frac{a_{1ij}(t)\Delta x}{\varepsilon}\right)-1\right), \quad (5.3.27)$$

$$(\phi_{ij})^2(\Delta y,\varepsilon,t) = \frac{\Delta y\varepsilon}{a_{2ij}(t)}\left(\exp\left(\frac{a_{2ij}(t)\Delta y}{\varepsilon}\right)-1\right). \quad (5.3.28)$$

Expanding equations (5.3.27) and (5.3.28) in Taylor series gives

$$(\phi_{ij})^2(\Delta x,\varepsilon,t) = \Delta x^2 + \mathcal{O}\left(\frac{a_{1ij}(t)\Delta x^3}{\varepsilon}\right),$$
$$(\phi_{ij})^2(\Delta y,\varepsilon,t) = \Delta y^2 + \mathcal{O}\left(\frac{a_{2ij}(t)\Delta y^3}{\varepsilon}\right),$$

respectively. In matrix notation the scheme (5.3.25)-(5.3.26) takes the form

$$U'(t) + A(t)U(t) = F(t), \quad (5.3.29)$$

where $A(t) \in \mathbb{R}^{(n-1)(m-1)} \times \mathbb{R}^{(n-1)(m-1)}$ and $U(t)$ and $F(t)$ are in $\mathbb{R}^{(n-1)(m-1)}$. The entries of $A(t)$ and $F(t)$ are given below.

$$
\begin{aligned}
A_{pp}(t) &= \frac{2\varepsilon}{(\phi_{ij})^2(\varepsilon,\Delta x,t)} + \frac{2\varepsilon}{(\phi_{ij})^2(\varepsilon,\Delta y,t)} + \frac{a_{1ij}(t)}{\Delta x} + \frac{a_{2ij}(t)}{\Delta y} + b_{ij}(t), \\
&\quad i = 1(1)n-1, \quad j = 1(1)m-1, \\
A_{p,p+1}(t) &= -\frac{\varepsilon}{(\phi_{ij})^2(\varepsilon,\Delta y,t)}, \qquad i = 1(1)n-1, \qquad j = 1(1)m-2, \\
A_{p,p-1}(t) &= -\frac{\varepsilon}{(\phi_{ij})^2(\varepsilon,\Delta y,t)} - \frac{a_{2ij}(t)}{\Delta y}, \quad i = 1(1)n-1, \qquad j = 2(1)m-1, \\
A_{p,p+(n-1)}(t) &= -\frac{\varepsilon}{(\phi_{ij})^2(\varepsilon,\Delta x,t)}, \qquad i = 1(1)n-2, \qquad j = 1(1)m-1, \\
A_{p,p-(n-1)}(t) &= -\frac{\varepsilon}{(\phi_{ij})^2(\varepsilon,\Delta x,t)} - \frac{a_{1ij}(t)}{\Delta x}, \quad i = 2(1)n-1, \qquad j = 1(1)n-1.
\end{aligned}
$$

$$
\begin{aligned}
F_p(t) &= f_p(t) + \left(\frac{\varepsilon}{(\phi_{11})^2(\varepsilon,\Delta x,t)} + \frac{a_{111}(t)}{\Delta x}\right) u(0,y_1,t) + \left(\frac{\varepsilon}{(\phi_{11})^2(\varepsilon,\Delta y,t)} + \frac{a_{211}(t)}{\Delta y}\right) \\
&\quad \times u(x_1,0,t), \\
F_p(t) &= f_p(t) + \left(\frac{\varepsilon}{(\phi_{1j})^2(\varepsilon,\Delta x,t)} + \frac{a_{11j}(t)}{\Delta x}\right) u(0,y_j,t), \quad j = 2(1)m-2, \\
F_p(t) &= f_p(t) + \left(\frac{\varepsilon}{(\phi_{1,m-1})^2(\varepsilon,\Delta x,t)} + \frac{a_{11,m-1}(t)}{\Delta x}\right) u(0,y_{m-1},t) + \left(\frac{\varepsilon}{(\phi_{1,m-1})^2(\varepsilon,\Delta y,t)}\right) \\
&\quad \times u(x_1,1,t), \\
F_p(t) &= f_p(t) + \left(\frac{\varepsilon}{(\phi_{i1})^2(\varepsilon,\Delta y,t)} + \frac{a_{2i1}(t)}{\Delta y}\right) u(x_i,0,t), \quad i = 2(1)n-2, \\
F_p(t) &= f_p(t), \qquad i = 2(1)n-2, \ \ j = 2(1)m-2, \\
F_p(t) &= f_p(t) + \left(\frac{\varepsilon}{(\phi_{i,m-1})^2(\varepsilon,\Delta y,t)}\right) u(x_i,1,t), \quad i = 2(1)n-2, \\
F_p(t) &= f_p(t) + \left(\frac{\varepsilon}{(\phi_{n-1,1})^2(\varepsilon,\Delta x,t)}\right) u(1,y_1,t) + \left(\frac{\varepsilon}{(\phi_{n-1,1})^2(\varepsilon,\Delta y,t)} + \frac{a_{2n-1,1}(t)}{\Delta y}\right) \\
&\quad \times u(x_{n-1},0,t), \\
F_p(t) &= f_p(t) + \left(\frac{\varepsilon}{(\phi_{n-1,j})^2(\varepsilon,\Delta x,t)}\right) u(1,y_j,t), \quad j = 2(1)m-2, \\
F_p(t) &= f_p(t) \left(\frac{\varepsilon}{(\phi_{n-1,m-1})^2(\varepsilon,\Delta x,t)}\right) u(1,y_{m-1},t) + \left(\frac{\varepsilon}{(\phi_{n-1,m-1})^2(\varepsilon,\Delta y,t)}\right) u(x_{n-1},1,t).
\end{aligned}
$$

Next we analyse the FOFDM for convergence.

5.3.2 Error analysis

The operator $\mathcal{L}^{\Delta x,\Delta y}$ as defined in the scheme (5.3.25)-(5.3.26) also satisfies the maximum principle and the uniform stability estimate. These properties are analogous to those found in the lemmas 5.2.1 and 5.2.2.

The truncation error of the scheme (5.3.25)-(5.3.26) is given by

$$\begin{aligned}\mathcal{L}^{\Delta x,\Delta y}(U_{ij}(t)-u_{ij}(t)) &= \frac{du_{ij}(t)}{dt}-\varepsilon\Delta u_{ij}(t)+a_{1ij}(t)(u_x(t))_{ij}+a_{2ij}(t)(u_y(t))ij\\ &\quad +b_{ij}(t)u_{ij}(t)-\left[\frac{du_{ij}(t)}{dt}-\varepsilon\left(\frac{u_{i+1}(t)-2u_i(t)+u_{i-1}(t)}{(\phi_{ij})^2(\varepsilon,\Delta x,t)}\right)\right.\\ &\quad -\varepsilon\left(\frac{u_{i,j+1}(t)-2u_{ij}(t)+u_{i,j-1}(t)}{(\phi_{ij})^2(\varepsilon,\Delta y,t)}\right)+a_{1ij}(t)\frac{u_{ij}(t)-u_{i-1,j}(t)}{\Delta y}\\ &\quad \left.+a_{2ij}(t)\frac{u_{ij}(t)-u_{i-1,j}(t)}{\Delta y}+b_{ij}(t)u_{ij}(t)\right]\\ &= -\varepsilon\Delta u_{ij}(t)+a_{1ij}(t)(u_x(t))_{ij}+a_{2ij}(t)(u_x(t))_{ij}\\ &\quad -\left(\frac{u_{i+1,j}(t)-2u_{ij}(t)+u_{i-1,j}(t)}{(\phi_{ij})^2(\varepsilon,\Delta x,t)}\right)\\ &\quad -\left(\frac{u_{i,j+1}(t)-2u_{ij}(t)+u_{i,j-1}(t)}{(\phi_{ij})^2(\varepsilon,\Delta y,t)}\right)+a_{1ij}(t)\frac{u_{ij}(t)-u_{i-1,j}(t)}{\Delta x}\\ &\quad +a_{2ij}(t)\frac{u_{ij}(t)-u_{i-1,j}(t)}{\Delta y}.\end{aligned}$$

Using appropriate Taylor series expansions yields

$$\begin{aligned}\mathcal{L}^{\Delta x,\Delta y}(U_{ij}(t)-u_{ij}(t)) &\leq -\varepsilon(u_{xx}(t))_{ij}-\varepsilon(u_{yy}(t))_{ij}+\left[\Delta x^2(u_{xx}(t))_{ij}+\frac{\Delta x^4}{12}(u_{xxxx}(t))_{ij}\right]\\ &\quad \times\frac{\varepsilon}{(\phi_{ij})^2(\varepsilon,\Delta x,t)}+\left[\Delta y^2(u_{yy}(t))_{ij}+\frac{\Delta y^4}{12}(u_{yyyy}(t))_{ij}\right]\\ &\quad \times\frac{\varepsilon}{(\phi_{ij})^2(\varepsilon,\Delta y,t)}+a_{1ij}(t)\left[\frac{\Delta x}{2}(u_{xx}(t))_{ij}-\frac{\Delta x^2}{6}(u_{xxx}(t))_{ij}\right]\\ &\quad a_{2ij}(t)\left[\frac{\Delta y}{2}(u_{yy}(t))_{ij}-\frac{\Delta y^2}{6}(u_{yyy}(t))_{ij}\right].\end{aligned}$$

Now using a truncated Taylor series expansion of the denominator function gives

$$\begin{aligned}\mathcal{L}^{\Delta x,\Delta y}(U_{ij}(t)-u_{ij}(t)) &\leq -\varepsilon(u_{xx}(t))_{ij}-\varepsilon(u_{yy}(t))_{ij}+\left(\frac{\varepsilon}{\Delta x^2}-\frac{a_{1ij}(t)}{\Delta x}+\frac{a_{1ij}^2(t)}{\varepsilon}\right.\\ &\quad \left.-\frac{a_{1ij}^3(t)\Delta x}{\varepsilon^2}\right)\left[\Delta x^2(u_{xx}(t))_{ij}+\frac{\Delta x^4}{12}(u_{xxxx}(t))_{ij}\right]\\ &\quad +\left(\frac{\varepsilon}{\Delta y^2}-\frac{a_{2ij}(t)}{\Delta y}+\frac{a_{2ij}^2(t)}{\varepsilon}-\frac{a_{2ij}^3(t)\Delta x}{\varepsilon^2}\right)\end{aligned}$$

$$
\begin{aligned}
&\times \left[\Delta y^2 (u_{yy}(t))_{ij} + \frac{\Delta y^4}{12}(u_{yyyy}(t))_{ij}\right] \\
&+ \left[\frac{a_{1ij}(t)}{2}\Delta x (u_{xx}(t))_{ij} - \frac{a_{1ij}(t)}{6}\Delta x^2 (u_{xxx}(t))_{ij}\right] \\
&+ \left[\frac{a_{2ij}(t)}{2}\Delta y (u_{yy}(t))_{ij} - \frac{a_{2ij}(t)}{6}\Delta y^2 (u_{yyy}(t))_{ij}\right] \\
= & -\frac{a_{1ij}^3(t)}{12\varepsilon^2}(u_{xxxx}(t))_{ij}\Delta x^5 + \frac{a_{1ij}^2(t)}{12\varepsilon}(u_{xxxx}(t))_{ij}\Delta x^4 \\
&+ \left[\frac{a_{1ij}(t)}{12}(u_{xxxx}(t))_{ij} - \frac{a_{1ij}^3(t)}{\varepsilon^2}(u_{xx}(t))_{ij}\right]\Delta x^3 \\
&+ \left[\frac{a_{1ij}^2(t)}{\varepsilon}(u_{xx}(t))_{ij} - \frac{a_{1ij}^2(t)}{6}(u_{xxx}(t))_{ij} + \frac{\varepsilon}{12}(u_{xxxx}(t))_{ij}\right]\Delta x^2 \\
&- \frac{a_{1ij}(t)\Delta x}{2}(u_{xx}(t))_{ij} - \frac{a_{2ij}^3}{12\varepsilon^2}(u_{yyyy}(t))_{ij}\Delta y^5 + \frac{a_{2ij}^2}{12\varepsilon}(u_{yyyy}(t))_{ij}\Delta y^4 \\
&+ \left[\frac{a_{2ij}}{12}(u_{yyyy}(t))_{ij} - \frac{a_{2ij}^3}{\varepsilon^2}(u_{yyy}(t))_{ij}\right]\Delta y^3 \\
&+ \left[\frac{a_{2ij}^2}{\varepsilon}(u_{yy}(t))_{ij} - \frac{a_{2ij}^2}{6}(u_{yyy}(t))_{ij} + \frac{\varepsilon}{12}(u_{yyyy}(t))_{ij}\right]\Delta y^2 \\
&- \frac{a_{2ij}\Delta y}{2}(u_{yy}(t))_{ij}.
\end{aligned}
$$

Application of the bound (5.1.3) and Lemma 2.3.1 gives

$$
||\mathcal{L}^{\Delta x, \Delta y}(U_{ij}(t) - u_{ij}(t))|| \;\leq\; C(\Delta x + \Delta y).
$$

From Lemma 5.2.2

$$
||(U_{ij}(t) - u_{ij}(t))|| \;\leq\; C(\Delta x + \Delta y),
$$

follows.

Lemma 5.3.1. *The error associated with the FOFDM satisfies*

$$
\max_{0<\varepsilon\leq 1}\max_{0\leq i\leq n; 0\leq j\leq m} ||(U_{ij}(t) - u_{ij}(t))|| \leq C(\Delta x + \Delta y).
$$

5.4 Time discretization

Here we integrate the semi-discrete problems (5.2.9)-(5.2.10) and (5.3.25)-(5.3.26) on the domain $(0, T]$ using the backward Euler on a uniform mesh. Now we write the fully descretized scheme as

$$\frac{U^k - U^{k-1}}{\tau} + A(t_k)U^k = F(t_k), \quad k = 1, ..., K, \tag{5.4.30}$$

with initial condition $U(0) = \varphi$.

The following theorem provides the main result of this chapter.

Theorem 5.4.1. *Let $u \in \mathcal{C}^{4,2}(\bar{Q})$ be the exact solution of the continuous problem (5.1.1)-(5.1.2) and U_{ij}^k be the numerical solution obtained via the FMDML (5.2.9)-(5.2.10) along with (5.4.30) or the FOFDML (5.3.25)-(5.3.26) along with (5.4.30). Then the errors of these methods are as follows:*

$$\sup_{0<\varepsilon\leq 1} \max_{0\leq i\leq n; 0\leq k\leq K} ||U_{ij}^k - u_{ij}^k|| \leq C(n^{-1}\ln n + \tau), \quad \textit{for the FMFDML}$$

and

$$\sup_{0<\varepsilon\leq 1} \max_{0\leq i\leq n; 0\leq j\leq m; 0\leq k\leq K} ||U_{ij}^k - u_{ij}^k|| \leq C(\Delta x + \Delta y + \tau), \quad \textit{for the FOFDML.}$$

5.5 Numerical example

In this section, we present a numerical example to support the theoretical findings. Similar to the earlier chapters, the exact solution of the problem is not known, thus to calculate the maximum pointwise error we use the formula

$$E_{n,\tau}^\varepsilon = \max_{0\leq i,j\leq n; 0\leq k\leq K} |U_{ij;n}^{k;K} - U_{ij;2n}^{k;2K}|. \tag{5.5.31}$$

Furthermore, we compute the rate of convergence with the formula

$$r_l^\varepsilon = \log_2\left(E_{n,\tau}^\varepsilon / E_{2n,\tau/2}^\varepsilon\right), \quad l = 1, 2, 3, ... \tag{5.5.32}$$

Example 5.5.1. *[10]*

$$u_t - \varepsilon\Delta u + \left(1 - \frac{xy}{2}\right)\frac{\partial u}{\partial x} + \left(1 + \frac{xy}{2}\right)\frac{\partial u}{\partial y} = e^{-t}t^2(x(1-x) + y(1-y)), \tag{5.5.33}$$

$$(x, y, t) \in \Omega \times (0, 1],$$

$$u(x, y, 0) = 0, \quad (x, y) \in \bar{\Omega}, \quad u(x, y, t) = 0, \quad \partial\Omega \times [0, 1]. \tag{5.5.34}$$

Table 5.1: Maximum pointwise error for Example 5.5.1 using the FMFDML

ε	$n = K = 8$	16	32
10^{0}	$3.58E-04$	$1.97E-04$	$1.04E-04$
10^{-1}	$4.29E-03$	$2.57E-03$	$1.46E-03$
10^{-2}	$8.69E-03$	$7.02E-03$	$5.89E-03$
10^{-3}	$9.14E-03$	$7.74E-03$	$7.35E-03$
10^{-4}	$8.40E-03$	$6.20E-03$	$5.66E-03$
10^{-5}	$7.92E-03$	$5.07E-03$	$3.73E-03$
10^{-6}	$7.73E-03$	$4.64E-03$	$2.83E-03$
10^{-7}	$7.66E-03$	$4.60E-03$	$2.58E-03$
10^{-8}	$7.64E-03$	$4.58E-03$	$2.54E-03$
10^{-9}	$7.63E-03$	$4.58E-03$	$2.53E-03$
10^{-10}	$7.63E-03$	$4.58E-03$	$2.53E-03$

Table 5.2: Maximum pointwise error for Example 5.5.1 using the FOFDML

ε	$n = K = 8$	16	32
10^{0}	$5.05E-05$	$2.82E-05$	$1.52E-05$
10^{-1}	$7.32E-04$	$4.75E-04$	$2.69E-04$
10^{-2}	$3.91E-03$	$2.00E-03$	$7.49E-04$
10^{-3}	$3.95E-03$	$2.28E-03$	$1.24E-03$
10^{-4}	$3.95E-03$	$2.28E-03$	$1.24E-03$
10^{-5}	$3.95E-03$	$2.28E-03$	$1.24E-03$
10^{-6}	$3.95E-03$	$2.28E-03$	$1.24E-03$
10^{-7}	$3.95E-03$	$2.28E-03$	$1.24E-03$
10^{-8}	$3.95E-03$	$2.28E-03$	$1.24E-03$
10^{-9}	$3.95E-03$	$2.28E-03$	$1.24E-03$
10^{-10}	$3.95E-03$	$2.28E-03$	$1.24E-03$

Table 5.3: Rate of convergence for Example 5.5.1 using the FMFDML

ε	r_1	r_2
10^0	0.8618	0.9225
10^{-1}	0.7180	0.7829
10^{-2}	0.4540	0.7148
10^{-3}	0.4779	0.7154
10^{-4}	0.4886	0.6970
10^{-5}	0.4878	0.6970
10^{-6}	0.4878	0.6970
10^{-7}	0.4878	0.6970
10^{-8}	0.4878	0.6970
10^{-9}	0.4878	0.6970
10^{-10}	0.4878	0.6970

Table 5.4: Rate of convergence for Example 5.5.1 using the FOFDML

ε	r_1	r_2
10^0	0.8400	0.8962
10^{-1}	0.6257	0.8192
10^{-2}	0.9638	1.4193
10^{-3}	0.7911	0.8808
10^{-4}	0.7911	0.8808
10^{-5}	0.7911	0.8808
10^{-6}	0.7911	0.8808
10^{-7}	0.7911	0.8808
10^{-8}	0.7911	0.8808
10^{-9}	0.7911	0.8808
10^{-10}	0.7911	0.8808

5.6 Conclusion

Two-dimensional convection-diffusion problems were considered. After inspected the FMFDML of [12], we investigated a FOFDML to integrate these problems. The FOFDML consists of a spatial discretization in both spatial variables followed by a temporal discretization using the backward Euler method. Error analyzes conducted showed that both methods are second order convergent in space (except for a logarithmic factor in the case of FMFDML) and first order in time. These theoretical findings are confirmed through simulations on a test example.

Chapter 6

Conclusion

To integrate numerically time-dependent problems, one can either resort to a step-by-step discretization (each independent variable at a time) or to a full discretization (all independent variables at once). Each strategy has its advantages and disadvantages. In this thesis, we used the method of lines (MOL) to solve singularly perturbed time-dependent problems. The MOL forms part of the step-by-step discretisation strategy. It consists of spatial discretization followed by the temporal one. The choice of the MOL was dictated by the relative ease in its analysis and implementation, like its counterpart where one starts with the discretization of the time variable. Further, for the MOL, the spatial discretization results in a system of initial value problems for which solvers are readily available and popular.

This dissertation is articulated around one- and two-dimensional time-dependent singularly perturbed problems. In either case, we studied the reaction-diffusion and the convection-diffusion problems.

Clavero and Jorge [12] studied the two-dimensional time-dependent singularly perturbed problems (both reaction-diffusion and convection-diffusion). They designed methods of lines based on fitted mesh finite difference methods (FMFDMs) for the space variable and the backward Euler method for the time variable. We refer to these methods as the fitted mesh finite difference methods of lines (FMFDMLs). We followed their idea to study the one-dimensional case using FMFDMLs. We also inspected the two-dimensional case.

As alternatives to these FMFDMLs, we developed methods which we refer to as the fitted operator finite difference methods of lines (FOFDMLs). They consist of spatial discretization via the fitted operator finite difference methods (FOFDMs) followed by the backward Euler method in the time variable. We adapted these new methods to each case (one- and two-dimensional, reaction-diffusion and convection-diffusion).

In Chapter 1, we introduced singularly perturbed problems (SPPs) and the types of finite difference methods that have been used to solve them. These are the FMFDMs and the FOFDMs. We also explained what the MOL entails. Then, we conducted a survey on the numerical methods for time-dependent SPPs. The chapter ended with the presentation of the objectives of this dissertation and that of its structure.

Chapter 2 deals with one-dimensional reaction-diffusion problems. The chapter begun by a presentation of some qualitative results regarding the exact solution and its derivatives. We proceeded by exploring a FMFDML through which the spatial variable is discretized on a piecewise uniform mesh (of Shishkin type). We showed that the system resulting from the spatial discretization enjoys a maximum principle which also implies that the discretization is stable. These properties of the spatial discretization guarantee that the numerical solution may replicate the exact solution in the limit. Analysis of the error of the FMFDML proved that the method is uniformly convergent of order almost two in space and of order one in time, with respect to the perturbation parameter.

Further, in this chapter, we designed and analyzed a FOFDML. The semi-discrete operator resulting from the spatial discretization uses an appropriately selected denominator function in accordance with the modelling rules of the nonstandard finite difference schemes. This semi-discrete operator also satisfies a maximum principle. As a result, the spatial discretization is stable. This fact is used in the convergence analysis. The FOFDML above was shown to be uniformly convergent of order two in space and one in time, with respect to the perturbation parameter.

Chapter 3 is concerned with one-dimensional convection-diffusion problems while chapters 4 and 5 treat the two-dimensional reaction-diffusion and convection-diffusion problems, respectively. All three chapters are structured in a similar manner as Chapter 2.

We are currently investigating the possibility of improving the accuracy of the methods presented in this dissertation. We also intend to extend these methods to nonlinear problems.

Bibliography

[1] J. R. Cash, Efficient time integrators in the numerical method of lines, *Journal of Computational and Applied Mathematics* **183(2)** (2005) 259–274.

[2] Y. Cheng, L. Lui, An Adaptive Grid Method for Singularly Perturbed Time-Dependent Convection-Diffusion Problems, *Communications in Computational Physics* **20(5)** (2016) 1340-1358.

[3] C. Clavero, J .C. Jorge, F. Lisbona, G. I Shishkin, A fractional step method on a special mesh for the resolution of multidimensional evolutionary convection-diffusion problems, *Applied Numerical Mathematics* **27(3)** (1998) 211-231.

[4] C. Clavero, J. C. Jorge, F. Lisbona, G.I. Shishkin, An alternating direction scheme on a nonuniform mesh for reaction-diffusion parabolic problems, *IMA Journal of Numerical Analysis* **20(2)** (2000) 263-280.

[5] C. Clavero, J. C. Jorge, F. Lisbona, A uniformly convergent scheme on a non-uniform mesh for convection–diffusion parabolic problems, *Journal of Computational and Applied Mathematics* **154(2)** (2003) 415-429.

[6] C. Clavero, J. Gracia, E. ORiordan, A parameter robust numerical method for a two dimensional reaction-diffusion problem, *Mathematics of Computation* **74(252)** (2005) 1743-1758.

[7] C. Clavero, J. L Gracia, A compact finite difference scheme for 2D reaction–diffusion singularly perturbed problems, *Journal of Computational and Applied Mathematics* **192(1)** (2006) 152–167.

[8] C. Clavero, J. L. Gracia and J. C Jorge, A uniformly convergent alternating direction HODIE finite difference scheme for 2D time-dependent convection–diffusion problems, *IMA Journal of Numerical Analysis* **26(1)** (2006) 155-172.

[9] C. Clavero, J. L. Gracia, On the uniform convergence of a finite difference scheme for time dependent singularly perturbed reaction-diffusion problems, *Applied Mathematics and Computation* **216(5)** (2010) 1478–1488.

[10] C. Clavero, J. L. Gracia, A HODIE method for 2D parabolic problems ofconvection-diffusion type, *Garcia de Galdeano* **27** (2003) 217-224.

[11] C. Clavero, J. L. Gracia, A high order HODIE finite difference scheme for 1D parabolic singularly perturbed reaction–diffusion problems, *Applied Mathematics and Computation* **218(9)** (2012) 5067-5080.

[12] C. Clavero, J. C. Jorge, Another uniform convergence analysis technique of some numerical methods for parabolic singularly perturbed problems, *IMA Journal of Numerical Analysis* **70(3)** (2015) 222-235.

[13] M. T. Diamantakis, The NUMOL solution of time-dependent PDEs using DESI RungeKutta formulae, *Applied Numerical Mathematics* **17(3)** (1995) 235–249.

[14] U. Erdoğan, K. Akarbulut, N. Ö. Tsan Exponentially Fitted Finite Difference Schemes for Reaction-Diffusion Equations, *Mathematical and Computational Applications* **31(3)** (2016) 32.

[15] A. El Hajaji, A numerical method for solving time-dependent convection-diffusion problems, *Boletim da Sociedade Paranaense de Matemática* **35(1)** (2015) 217-228.

[16] P. A. Farrell, A. F. Hegarty, J. J. H. Miller, E. O'Riordan, G. I. Shishkin, Robust computational techniques for boundary layers, CRC Press(2000).

[17] I. Babuška et al., SIAM Monographs on Mathematical Modelling and Computation, SIAM (1997).

[18] S. Gottlieb, D. I. Ketcheson, C-W Shu, Strong stability preserving Runge-Kutta and multistep time discretizations, World Scientific Singapore(2011).

[19] J. L Gracia, F. J.Lisbona, A uniformly convergent scheme for a system of reaction–diffusion equations, *Journal of Computational and Applied Mathematics* **206(1)** (2007) 1-16.

[20] L. Graney, A. A. Richardson, The numerical solution of non-linear partial differential equations by the method of lines, *Journal of Computational and Applied Mathematics* **7(4)** (1981) 229–236.

[21] M. K. Kadalbajoo, A. Awasthi, A parameter uniform difference scheme for singularly perturbed parabolic problem in one space dimension, *Applied Mathematics and Computations* **183(1)** (2006) 42-60.

[22] M. K. Kadalbajoo, V. Gupta, A. Awasthi, A uniformly convergent B-spline collocation method on a nonuniform mesh for singularly perturbed one-dimensional time-dependent linear convection–diffusion problem, *Journal of Computational and Applied Mathematics* **220(1)** (2008) 271-289.

[23] N. Kopteva, S. B. Savescu, Pointwise error estimates for a singularly perturbed time-dependent semilinear reaction–diffusion problem,*IMA Journal of Numerical Analysis*(2010) drp032.

[24] O. A Ladyzhenskaia, V. A. Solonnikov, N. N Ural'tseva, Linear and quasi-linear equations of parabolic typeAmerican Mathematical Soc.**23**,(1988)

[25] W. Lenferink, A second order scheme for a time-dependent, singularly perturbed convection–diffusion equation, *Journal of Computational and Applied Mathematics* **143(1)** (2002) 49-68.

[26] T. Linß, M. Stynes, A hybrid difference scheme on a Shishkin mesh for linear convection–diffusion problems, *Applied numerical mathematics* **31(3)** (1999) 255–270.

[27] T. Linß, Layer-adapted meshes for reaction-convection-diffusion problems, *Springer* (2009) .

[28] T. Linß, Layer-adapted meshes and FEM for time-dependent singularly perturbed reaction-diffusion problems, *International Journal of Computing Science and Mathematics* **1(2-4)** (2007) 259-270.

[29] T. Linß, N. Madden, Parameter uniform approximations for time-dependent reaction-diffusion problems, *Numerical Methods for Partial Differential Equations* **23(6)** (2007) 1290-1300.

[30] J.M.-S. Lubuma, K.C. Patidar, Uniformly convergent non-standard finite difference methods for self-adjoint singular perturbation problems, *Journal of Computational and Applied Mathematics* **191** (2006) 228-238.

[31] J.M.-S. Lubuma, K.C. Patidar, Non-standard methods for singularly perturbed problems possessing oscillatory/layer solutions, *Applied Mathematics and Computation* **187** (2007) 1147-1160.

[32] R.E. Mickens, *Non-standard Finite Difference Models of Differential Equations*, World Scientific, Singapore, 1994.

[33] R.E. Mickens, Calculation of denominator functions for non-standard finite difference schemes for differential equations satisfying a positivity condition, *Numerical Methods for Partial Differential Equations* **23(3)** (2007) 672-691.

[34] J. J. H. Miller, E. ORiordan, G.I. Shishkin, Fitted numerical methods for singularly perturbed problems. Error Estimates in the maximum norm for linear problems in one and two dimension, *World Scientific Publications, Singapore*(2012).

[35] J. J. H. Miller, E. O'Riordan, G. I. Shishkin, L. P. Shishkina, Fitted mesh methods for problems with parabolic boundary layers, *Mathematical proceedings-Royal Irish Academy* **98** (1998) 173-190.

[36] J. J. H. Miller, E. O'Riordan, G. I. Shishkin, B. Koren, L. P. Shishkina, Grid approximation of a singularly perturbed boundary value problem modelling heat transfer in

the case of flow over a flat plate with suction of the boundary layer, *Computational and applied Mathematics* **166(1)** (2004) 221-232.

[37] K. Mukherjee, S. Natesan, Richardson extrapolation technique for singularly perturbed parabolic convection-diffusion problems, *Computing* **92(1)**(2011) 1-32.

[38] J. B. Munyakazi, Higher order numerical methods for singular perturbation problems, PhD thesis, *University of the Western Cape* (2009).

[39] J. B. Munyakazi, A Robust Finite Difference Method for Two-Parameter Parabolic Convection-Diffusion Problems, *Appl. Math* **9(6)** (2015) 2877–2883.

[40] J. B. Munyakazi, K. C. Patidar, Higher order numerical methods for singularly perturbed elliptic problems, *Neural, Parallel & Scientific Computations* **18(1)** (2010) 75-88.

[41] J. B. Munyakazi, K. C. Patidar, Limitations of Richardsons extrapolation for a high order fitted mesh method for self-adjoint singularly perturbed problems, *Journal of Applied Mathematics and Computing* **32(1)** (2010) 219-236.

[42] J. B. Munyakazi, K. C. Patidar, Novel fitted operator finite difference methods for singularly perturbed elliptic convection–diffusion problems in two dimensions, *Journal of Difference Equations and Applications* **18(5)** (2012) 799-813.

[43] J. B. Munyakazi, K. C. Patidar, A fitted numerical method for singularly perturbed parabolic reaction-diffusion problems, *Computational and Applied Mathematics* **32(3)** (2013) 509-519.

[44] S. Natesan, S. Gowrisankar, Robust numerical scheme for singularly perturbed parabolic initial-boundary-value problems on equidistributed mesh, *Computer Modelling in Engineering & Sciences(CMES)* **88(4)** (2012) 245-267.

[45] M. J. Ng-Stynes, E. O'Riordan, M. Stynes, Numerical methods for time-dependent convection-diffusion equations, *Journal of Computational and Applied mathematics* **21(3)** (1988) 289-310.

[46] S. Natesan, R. Deb, A Robust Numerical Scheme for Singularly Perturbed Parabolic Raction-diffusion Problems, *Neural, Parallel and Scientific Computations* **16(3)** (2008) 419.

[47] R. E. O'Malley, On the asymptotic solution of the singularly perturbed boundary value problems posed by Bohé, *Journal of Mathematical Analysis and Applications* **242(1)** (2000) 18-38.

[48] E. O'Riordan, M. Stynes, A globally uniformly convergent finite element method for a singularly perturbed elliptic problem in two dimensions, *Mathematics of Computation* **57(195)** (1991) 47-62.

[49] K. C. Patidar, K. K. Sharma, ε-uniformly convergent non-standard finite difference methods for singularly perturbed differential difference equations with small delay, *Applied Mathematics and Computation* **175(1)** (2006) 864-890.

[50] M. L. Peña, Asymptotic expansion for the initial value problem of the sunflower equation, *Journal of Mathematical Analysis and Applications* **143(2)** (1989) 471-479.

[51] L. Prandtl, "Uber Flussigkeitsbewegung bei sehr kleiner Reibung" Proc. III Intern. Congr. Math., Heidelberg, (1904).

[52] J. I. Ramos, An exponentially fitted method for singularly perturbed one-dimensional, parabolic problems *Applied Mathematics and Computation* **161** (2005) 513-532.

[53] S. C. S. Rao, V. Srivastava, Parameter-Robust Numerical Method for Time-Dependent Weakly Coupled Linear System of Singularly Perturbed Convection-Diffusion Equations, *Differential Equations and Dynamical Systems* (2016) 1-25.

[54] K. Rektorys The Method of Discretization in Time and Partial Differential Equations, *Equadiff 5* (1982) 293–296.

[55] H. Roos, M. Stynes, L. Tobiska Robust numerical methods for singularly perturbed differential equations: convection-diffusion-reaction and flow problems, Springer Science & Business Media **24** (2008).

[56] M. Sadiku, C. Obiozor, A simple introduction to the method of lines, *International Journal of Electrical Engineering Education* **37(3)** (2000) 282-296.

[57] M. Stynes, The midpoint upwind scheme, *Applied Numerical Mathematics* **32** (1997) 291-308.

[58] M. Stynes, Steady-state convection-diffusion problems, *Acta Numerica* **14** (2005) 445-508.

[59] K. Surla, V. Jerkovic, Some possibilities of applying spline collocations to singular perturbation problems, *Numerical Methods and Approximation Theory* **2** (1985) 19-25.

[60] S. Valarmathi, N. Ramanujam, An asymptotic numerical method for singularly perturbed third-order ordinary differential equations of convection-diffusion type, *Computers & Mathematics with Applications* **44(5)** (2002) 693-710.

[61] A. B. Vasil'eva, Asymptotic behaviour of solutions to certain problems involving nonlinear differential equations containing a small parameter multiplying the highest derivatives, *Russian Mathematical Surveys* **18(3)** (1963) pages 13.

[62] A. B. Vasil'eva, V. F. Butuzov, L. V. Kalachev, The boundary function method for singular perturbation problems, *SIAM* **14** (1995).

[63] G. Zhou, B. Wu, A variational-perturbation method for solving the time dependent singularly perturbed reaction diffusion problems, *Thermal Science* **20(3)** (2016) S801-S804.