Elementary Logic as a Tool in Proving Mathematical Statements

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A thesis submitted in partial fulfillment of the requirements for the degree of Magister Scientiae in the Department of Mathematics, University of the Western Cape.

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Supervisor: Prof. R.L. Fray

November 2008

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Romans 11:36
"For of Him, and through Him, and for Him are all things: to Him be glory forever."

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Abstract

An analysis of South African school mathematics results indicates that one of the problem areas in the mathematical performance of learners is proof and proving. In an endeavour to improve the mathematical proving ability of first year students at UWC, the MAM 112 class (a first year elective mathematics course) was taught a course in elementary logic.

In the initial part of the study, logic puzzles were utilized as a tool to teach students to make logical connections between and from mathematical statements using the rules of inference. Subsequently research was done to determine if knowledge and understanding of logic would translate into improved proving abilities of students.

To put proof and proving into perspective the origins and functions of proof was explicated and proving in South African schools was investigated. Consequently reasons are advanced for the dismal high school mathematics results in terms of proof and possible solutions are discussed.

Recent discoveries of neuroscience are utilized to delineate the brain structures and cognitive processes involved in learning so as to gain a better understanding of the learning of mathematics. The findings of neuroscience, cognitive psychology and educational psychology are employed to elucidate the influence of emotion, confidence, experience and practice on the learning of mathematics in order to determine which factors can be applied to improve the proving abilities of students.

The findings of the study indicate that knowledge of logic does help to improve the ability of students to make logical connections (deductions) between and from statements. The results of the study, however, do not indicate that knowledge and understanding of logic translates into improved proving ability of mathematical statements by students.

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Declaration

I declare that

Elementary Logic as a Tool in Proving Mathematical Statements

Is my own work, that it has not been submitted for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Bruce Matthew May November 2008

……………………………..

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Acknowledgements

The following people have earned my eternal gratitude for their help, support, encouragement and guidance in the completing of this thesis.

Praise be to my Lord and saviour Jesus Christ the source of my strength and without whom this thesis would not have been possible.

I will be forever grateful to Prof. R. L. Fray who has presented me with this opportunity and who with gentle wisdom has guided me patiently.

The Lord sometimes sends people on your path to lighten your load. Prof. R. Madsen is an example of such a person, his help with the statistical analysis of the data has proven to be invaluable.

Prof. R. Blignaut has helped to interpret the statistical results. She has done this unselfishly and without thought of personal gain.

My wife who has sacrificed much has been a pillar of strength and a source of excellent advice and counsel throughout the research journey.

I will be forever thankful to my mother for her constant prayers, advice, motivation and unfailing belief in my abilities.

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CHAPTER 1

INTRODUCTION

1.1 The influence of Mathematics on Economic Development

The development of good Mathematics and Science teachers and students has been identified by many as a prerequisite for economic development in countries. The South African government for example has identified Mathematics, Science and Technology as areas in education that need investment as a prerequisite for economic growth. The Dinaledi[1] project is an example of such investment by the South African government. Naledi Pandor (South African minister of education) had the following to say about the importance of these subjects: "Maths, science and technology are now more important than they have been in our recorded history." She went on to say that the importance of these subjects was also highlighted in the Accelerated and Shared Growth Initiative for South Africa (Asgi-SA)[2] and that Asgi-SA aims to increase economic growth to 6% per annum between 2010 and 2014, while also halving unemployment and poverty by 2014. In order to achieve this, critical skills and sectors have been identified. These include the skills of engineers and the information communication technology (ICT) sector, which both require a strong knowledge base of mathematics and science. There are however a number of factors that can and will influence the success or otherwise of such intervention by government.

1.2 Background to the study

The TIMSS [3] **[47]** report of 2003 indicates that South African grade 8 learners had the lowest performance in mathematics and science of the 50 countries that participated in the study. The study found that in mathematics, South African learners performed relatively well in the domains of measurement and data, while scoring the *lowest in geometry*.

The problems that learners experience in mathematics in South Africa is not restricted to grade 8, but surfaces as early as in Primary school as can be seen from the following two

[1] A crucial initiative arising from the National Strategy for Mathematics, Science and Technology Education in South Africa was the establishment of the Dinaledi Project in June 2001. As a result of this project, 102 secondary schools were selected to be centers of excellence for the development of mathematics, science and technology and was aimed at increasing the participation rates of especially previously disadvantaged and girl learners and to improve learner performance in these subjects.

[2] AsgiSA is a set of government interventions which seek to achieve an average economic growth of 6% by 2010 and to halve poverty and unemployment by 2014.

[3] TIMSS (Trends in International Mathematics and Science Study) – is a large-scale comparative study conducted internationally at the end of the grade 4 and grade 8 year; TIMSS primarily measures learner achievement in mathematics and science; TIMSS is a project of the International Association for the Evaluation of International Achievement (IEA); The Human Sciences Research Council has coordinated and managed the South African part of the study; South Africa was one of 50 countries that participated in the study; the study was done over four years and 9 000 grade 8 learners from South African schools participated in the study.

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news paper articles[4]. The first article (shown in figure 1) is an article that appeared in the Cape Argus on 22 May 2007 and the second article appeared in the same newspaper on 23 April 2008. The first article reports on a study that was done by the Western Cape Education Department (W.C.E.D.). This study was done at 1086 schools and 82 879 learners participated. The study measured the achievement in numeracy and literacy of grade 3 learners. The study found that approximately 70% of the grade 3 learners failed to meet the national curriculum requirements for numeracy. The second article reports on the testing (also done by W.C.E.D.) of 71 847 grade 6 learners in numeracy and literacy. The learners were tested for ability in numeracy and literacy. The report show that only 14% of grade 6 learners achieved more than 50% in numeracy.

[4]The statistics used in the articles is published by the department of education in their Education Management Information Systems (EMIS) of February 2008

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70% of province's Grade 3s can't do maths, says study

By PETER LUHANGA
Special Correspondent

Almost 70% of Grade 3 pupils in the Western Cape cannot do maths and 50% cannot read and write to the standards required of them by the national curriculum.

Shocking figures revealed after a study done by the Western Cape Education Department showed that 69% of Grade 3 pupils in the province failed to meet national curriculum requirements for numeracy, while over 50% did not meet literacy requirements.

Although the figures on literacy showed a marginal improvement on previous years, numeracy appeared to be continuing to decline.

Education specialists said the problem was clearly linked to teachers' knowledge of numeracy and the way it should be taught. Theyalso pointed out that parents' lack of numeracy skills played a role in their children's shortcomings.

To page 3

'70% of Grade 3s can't do maths'

From page 1

A total of 82 879 pupils at 1 086 schools took part in this latest study, conducted in October and November 2006, which looked at the extent to which pupils met the requirements of the national curriculum for Grade 3 literacy and numeracy.

Previous studies of Grade 3s were done in 2002 and 2004.

In a statement by Western Cape head of education Ron Swartz, it was revealed that the pass rate for Grade 3 numeracy pupils was 31%, reflecting a decline of 6.5% from 37.3% in previous studies.

The pass rate for Grade 3 pupils in literacy improved from 39.5% in 2004 to 47.7% in 2006, an increase of 8.2% and an overall improvement of 12.2% over the three testing periods.

While noting that the literacy improvements were encouraging, Swartz said 47.7% was still too low and that the department would continue to implement their literacy and numeracy strategy to ensure continued improvement.

He said the literacy and numeracy strategy, introduced in July 2006, included the deployment of 510 teaching assistants to assist Grade 1 to 3 teachers in 163 primary schools in poor areas.

The department was less happy with the results of the numeracy study. "It seems clear that the drop in numeracy can be attributed to teachers' knowledge of the learning area, as well as the various and appropriate methodologies to teach numeracy more effectively," Swartz said.

"This will obviously form the basis of our interventions in this regards."

Officials from the department were currently visiting schools to discuss the results.

Contacted about the results, Robert Prince, director of UCT's numeracy centre, said the numeracy levels of Grade 3 pupils were affected by various factors, ranging from teachers' knowledge of numeracy to the practice of teaching it.

He said teacher training and re-training programmes were essential ways to improve teacher knowledge.

Prince said an area that was poorly understood was the role that parents' knowledge of numeracy could play in developing child numeracy.

He said the introduction of mathematical literacy teaching in Grades 10 to 12 should contribute to an improvement in the numeracy knowledge of future parents. – West Cape News

Figure 1

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Maths crisis in primary school

Poor kids do worst, says Dugmore

By CANDES KEATING
Education Reporter

The majority of Grade 6 pupils in Western Cape schools cannot do maths, while more than 50% perform below the standard of reading or writing required by the national curriculum.

Figures released by the Western Cape Education Department show that maths performance at primary schools is in decline.

The figures, given in a report on Grade 6 literacy and numeracy, show that only 14% of 11-year-olds achieved more than 50% in numeracy.

The report was based on the testing of 71 847 Grade 6 pupils in the province last year.

In 2005, 17.2% of the Grade 6 pupils tested for numeracy passed.

This was 3.2% higher than the percentage last year.

Education MEC Cameron Dugmore said: "It is very clear that maths remains a huge challenge for us in the province.

"That we had a marginal decline from 2005 underlines the need for us to strengthen the numeracy programme."

Dugmore said results for numeracy were lowest at poor schools.

The pass rate at poor schools was as low as 1.7%, with wealthier schools attaining 34.8%.

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"The latest study continues to reflect the huge differences between learners from different communities and the huge task confronting us as we seek to provide access to quality education for all," he said.

Dugmore said there were "no quick fixes" to the numeracy problem, but the department would increase support to poor schools.

Meanwhile, the deputy director-general for curriculum management, Brian Schreuder, said the literacy result had increased from 42.1% in 2005 to 44.8%.

"This is significant. This is something that we can build on," he said.

At poor schools, however, the literacy pass rate was 22%, while that among pupils at wealthier schools was 71.7%.

The medium of instruction was also a determining factor in results at some schools.

At eight schools participating in the department's language transformation programme, where pupils were taught in their mother tongue, results had improved significantly.

These schools opted to write the test in Xhosa and saw their pass rate double.

Schreuder said, however: "The overall results are not what we want."

Now that its literacy and numeracy strategy was in place at schools, the department would look at strengthening aspects of it, Schreuder said.

The provincial secretary of the South African Democratic Teachers' Union, Jonovan Rustin, blamed the results at poor schools on the large sizes of classes and lack of resources.

"The first thing that has an impact on these results is the class sizes. Many rich schools can afford to employ teachers in governing body posts and reduce class sizes. Rich schools can also buy additional resources."

Also, large numbers of qualified maths teachers were leaving poor schools to take up posts at rich schools.

"We have to start attracting more maths teachers to poor schools and drastically reduce class sizes if we want to improve results," Rustin said.

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Figure 2

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Learners are taught proofs in mathematics without being taught the different elements that constitute a proof and the different types of proof. The result is that whenever the learners are required to do a proof that they have not rote learned they normally struggle. This is especially so in the case of Euclidean geometry, (which forms the bulk of the proving experiences of learners) where learners have to substantiate their arguments but since they do not fully understand the proofs involved, they perform very badly in this section of the work. So it seems that the problems learners had with geometry in grade 8 continue through to the senior grades. This is verified by the fact that in most questions where learners are required to prove mathematical statements they perform very badly as shown in the analysis of the grade 12 external exams of 2004, 2005 and 2006 of the Western Cape Education Department (WCED). The questions and question analysis of 2004 is shown in figure 3 and 4; the questions and question analysis of 2005 is presented in figure 5 and 6; the questions and question analysis of 2006 is shown in figure 7 and 8.

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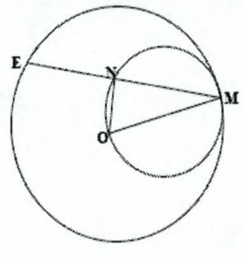
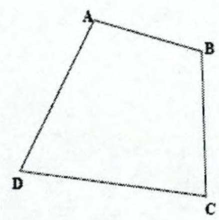
QUESTION 7

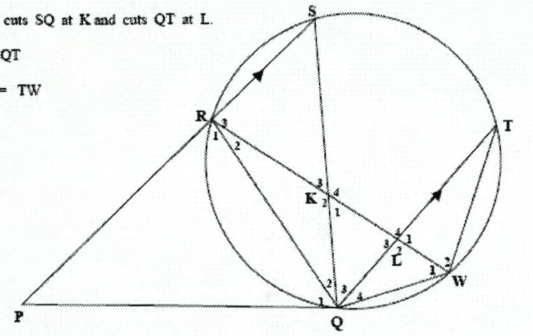
7.1 In the diagram alongside, OM is the radius of the larger circle and also the diameter of the smaller circle. Chord EM of the larger circle cuts the smaller circle at N.
If EM = $(2x^2 - 2)$ units and
ON = $2x$ units,
express, giving reasons, the length of the radius of the larger circle in terms of x. (6)



7.2 In the diagram alongside, ABCD is a quadrilateral such that $\hat{B} + \hat{D} = 180°$.
Use the diagram on the diagram sheet or redraw the diagram in your answer book, to prove the theorem which states that ABCD is a cyclic quadrilateral. (6)



7.3 In the diagram alongside, PQ is a tangent at Q.
PRS is a secant of circle RSTWQ.
RW cuts SQ at K and cuts QT at L.
$PS \parallel QT$
RS = TW



Prove that:

7.3.1 KQ is a tangent to circle LQW (5)

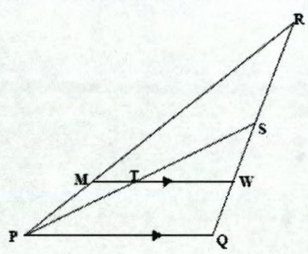
7.3.2 $\hat{R}_1 = \hat{L}_3$ (4)

7.3.3 PRKQ is cyclic quadrilateral (4)

7.3.4 RSLQ is a not a cyclic quadrilateral (2)
[27]

QUESTION 8

In the diagram alongside,
PS is a median of Δ PRQ.
T is the midpoint of PS
and $MTW \parallel PQ$.



Calculate the numerical value of the following:

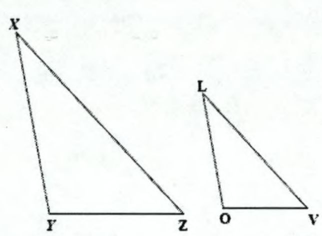
8.1 $\frac{RM}{RP}$ (6)

8.2 $\frac{\text{area } \Delta \text{ RPS}}{\text{area } \Delta \text{ RMW}}$ (4)
[10]

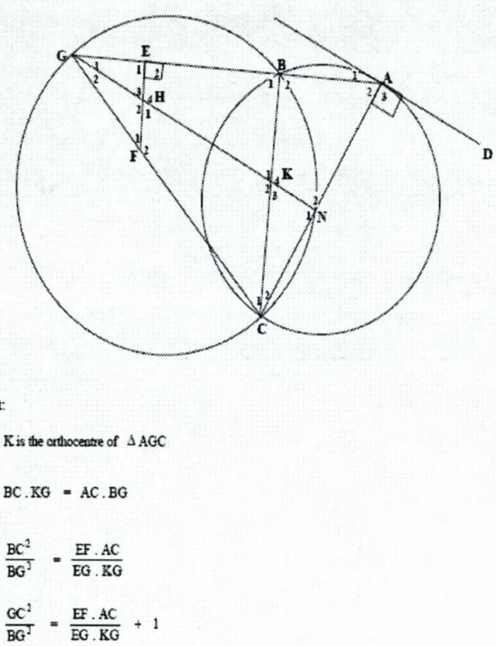
QUESTION 9

9.1 In the diagram alongside,
$\hat{X} = \hat{L}$, $\hat{Y} = \hat{O}$ and $\hat{Z} = \hat{V}$
Use the diagram on the diagram sheet, or redraw the diagram in your answer book to prove the theorem which states that

$$\frac{XY}{LO} = \frac{XZ}{LV}$$ (7)



9.2 In the diagram below, BC is a common chord of circles BCG and ABC.
AD is a tangent to circle ABC at A.
GHKN and GEBA are straight lines.
AD $\perp$ AC and FE $\perp$ GA



Prove that:

9.2.1 K is the orthocentre of Δ AGC (6)

9.2.2 BC . KG = AC . BG (5)

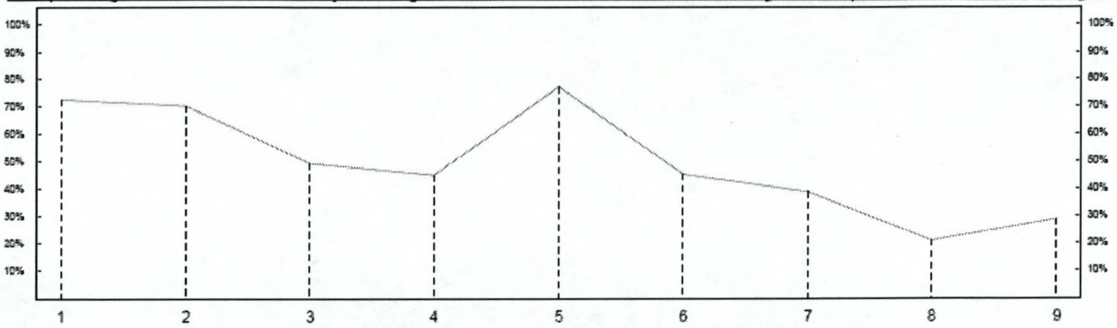
9.2.3 $\frac{BC^2}{BG^2} = \frac{EF \cdot AC}{EG \cdot KG}$ (6)

9.2.4 $\frac{GC^2}{BG^2} = \frac{EF \cdot AC}{EG \cdot KG} + 1$ (4)
[28]

Figure 3
SOURCE: Mathematics HG paper 2 – Nov 2004 (Department of Education RSA)

VRAAGONTLEDING / QUESTION ANALYSIS - NOVEMBER 2004

Verspreiding van Gemiddelde Punte per Vraag as Persentasies / Distribution of Average Marks per Question as Percentages



Gemiddelde Persentasie per Vraag / Average Percentage per Question

Question:	1	2	3	4	5	6	7	8	9	Paper
Max mark:	24	25	23	24	18	21	27	10	28	200
Average WCED:	72.5	70.3	49.3	45.0	77.2	45.3	39.0	21.0	28.8	50.0
No candidates:	5050	5047	5052	5049	5038	4986	5003	4506	4781	5062

SOURCE: WCED Examination Results 2004

Figure 4

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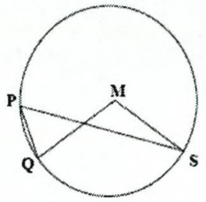
QUESTION 7

7.1 In the diagram alongside, circle PQS is drawn.

Use the diagram on the diagram sheet or redraw the diagram in your answer book to prove the theorem which states that:

If M is the centre of circle PQS, then $\hat{M} = 2\hat{P}$ (6)



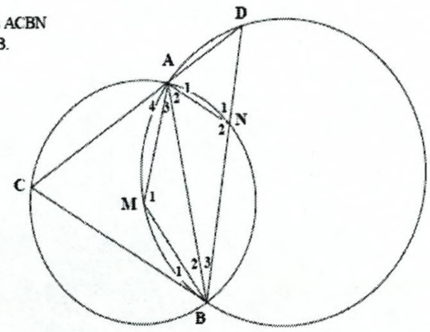
7.2 In the diagram alongside, circles ACBN and AMBD intersect at A and B.

CB is a tangent to the larger circle at B.

M is the centre of the smaller circle.

CAD and BND are straight lines.

Let $\hat{A}_3 = x$



7.2.1 Determine the size of $\hat{D}$ in terms of x. (4)

7.2.2 Prove that:

(a) CB || AN (7)

(b) AB is a tangent to circle ADN. (4)

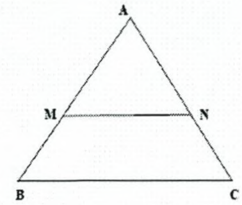
[21]

QUESTION 8

8.1 In the diagram alongside, M and N as two points on AB and AC respectively of ? ABC.

Use the diagram on the diagram sheet or redraw the diagram in your answer book to prove the theorem which states that:

If $\frac{AM}{MB} = \frac{AN}{NC}$, then MN || BC. (7)

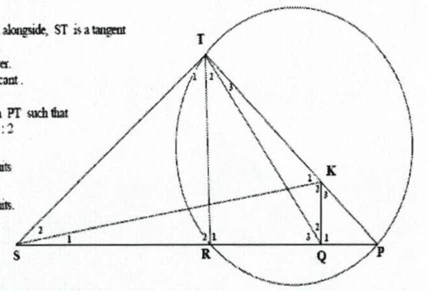


8.2 In the diagram alongside, ST is a tangent to circle TRP.
PT is a diameter.
SRQP is a secant.

K is a point on PT such that PK : KT = 1 : 2

PR = $\sqrt{18}$ units

PQ = $\sqrt{2}$ units.



8.2.1 Prove that:

(a) RT || QK (4)

(b) TKQS is a cyclic quadrilateral. (5)

(c) Δ QRT ||| Δ KTS (4)

8.2.2 If PS = $\sqrt{32}$ units, calculate stating reasons and **without using a calculator**, the length of:

(a) ST (6)

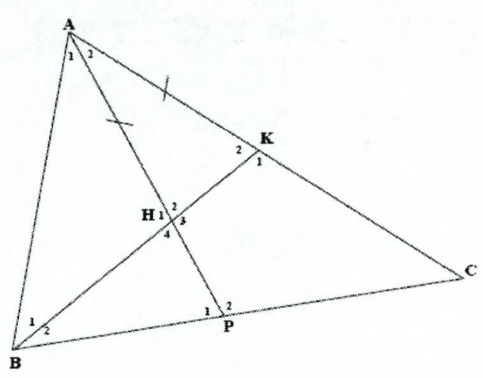
(b) KT (5)

[31]

QUESTION 9

In the diagram below, $A\hat{B}C$ is bisected by BK with K on AC.

AP and BK intersect at H with P on BC so that AH = AK



9.1 Prove that $\frac{AB}{BC} = \frac{AK}{KC}$ (7)

9.2 If it is further given that AKPB is a cyclic quadrilateral and that H is the centre of the inscribed circle of ΔABC, calculate the size of $B\hat{A}C$, stating reasons. (7)

[14]

Figure 5

SOURCE: Mathematics HG paper 2 – Nov 2005 (Department of Education RSA)

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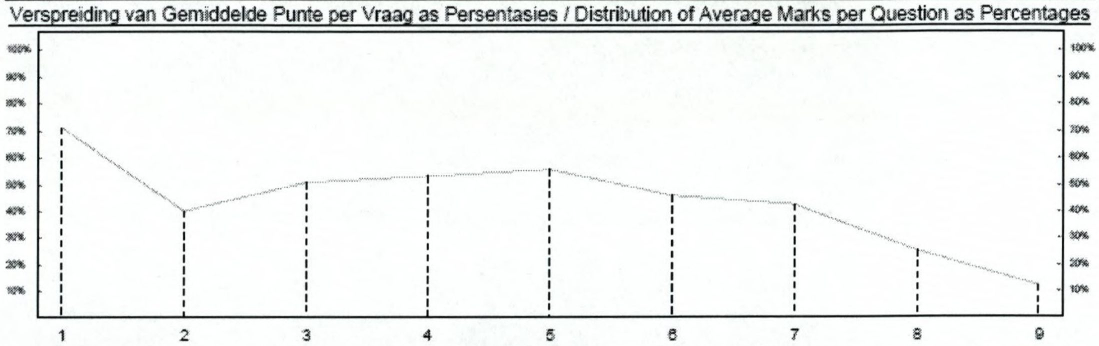
VRAAGONTLEDING / QUESTION ANALYSIS - NOVEMBER 2005

Verspreiding van Gemiddelde Punte per Vraag as Persentasies / Distribution of Average Marks per Question as Percentages



Gemiddelde Persentasie per Vraag / Average Percentage per Question

Question:	1	2	3	4	5	6	7	8	9	Paper
Max mark:	26	24	18	16	25	25	21	31	14	200
Average WCED:	70.7	39.8	50.4	52.7	55.2	45.6	42.2	24.9	11.7	44.0
No candidates:	4992	4975	4981	4972	4977	4955	4915	4843	4045	4994

SOURCE: WCED Examination Results 2005

Figure 6

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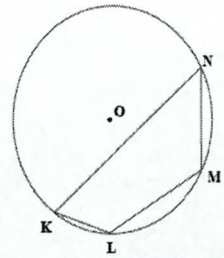
QUESTION 7

7.1 In the diagram below, circle KLNM is drawn.

Use the diagram on the diagram sheet or redraw the diagram in your answer book to prove the theorem which states that:

If O is the centre of the circle, then $\hat{L} + \hat{N} = 180°$



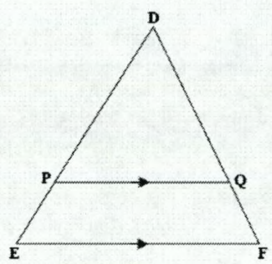
(6)

QUESTION 8

8.1 In the diagram alongside, P is a point on DE and Q is a point on DF of Δ DEF.

Use the diagram on the diagram sheet or redraw the diagram in your answer book to prove the theorem which states that:

If PQ $\parallel$ EF, then $\frac{DP}{DE} = \frac{DQ}{DF}$



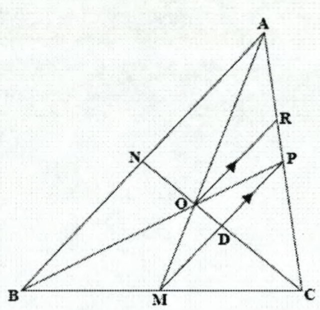
(7)

8.2 In the diagram alongside, medians AM and CN of Δ ABC, intersect at O.

BO is produced to cut AC at P.

MP and CN intersect at D.

OR $\parallel$ MP with R on AC.

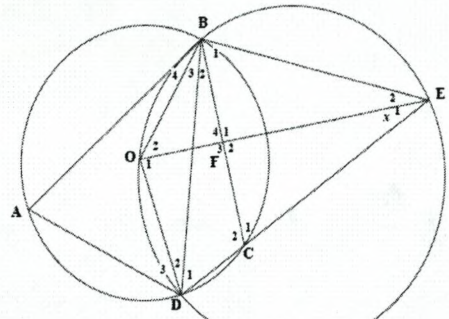


8.2.1 Calculate, with reasons, the numerical value of $\frac{ND}{NC}$ (6)

8.2.2 Use AO : AM = 2 : 3, to calculate the numerical value of $\frac{RP}{PC}$ (5)

[18]

7.2 In the diagram below, O is the centre of circle ABCD. DC is produced to meet circle BODE at E. OE intersects BC at F.



Let $\hat{E}_1 = x$

7.2.1 Determine the size of $\hat{A}$ in terms of x. (6)

7.2.2 Prove that:

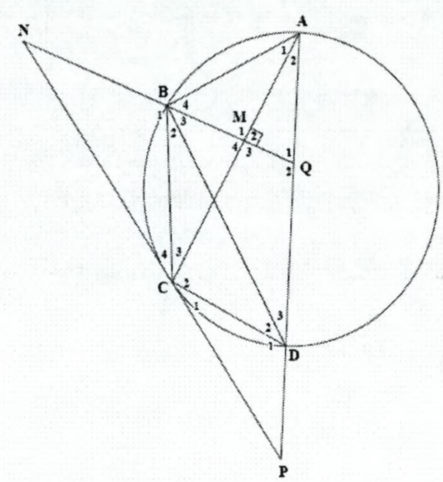
(a) BE = EC (7)

(b) BE is NOT a tangent to circle ABCD (3)

[22]

QUESTION 9

In the diagram below AD is the diameter of circle ABCD.
AD is produced to meet tangent NCP at P.
Straight line NB is produced to Q and intersects AC at M with Q on ADP.
AC $\perp$ NQ at M.



9.1 Prove that NQ $\parallel$ CD. (3)

9.2 Prove that ANCQ is a cyclic quadrilateral. (4)

9.3 9.3.1 Prove that Δ PCD $|||$ Δ PAC (3)

9.3.2 Hence, complete the statement: $PC^2 = \ldots$ (2)

9.4 Prove that $BC^2 = CD \cdot NB$ (7)

9.5 If it is further given that PC = MC, prove that

$$1 - \frac{BM^2}{BC^2} = \frac{AP \cdot DP}{CD \cdot NB}$$ (5)

[24]

Figure 7

SOURCE: Mathematics HG paper 2 – Nov 2006 (Department of Education RSA)

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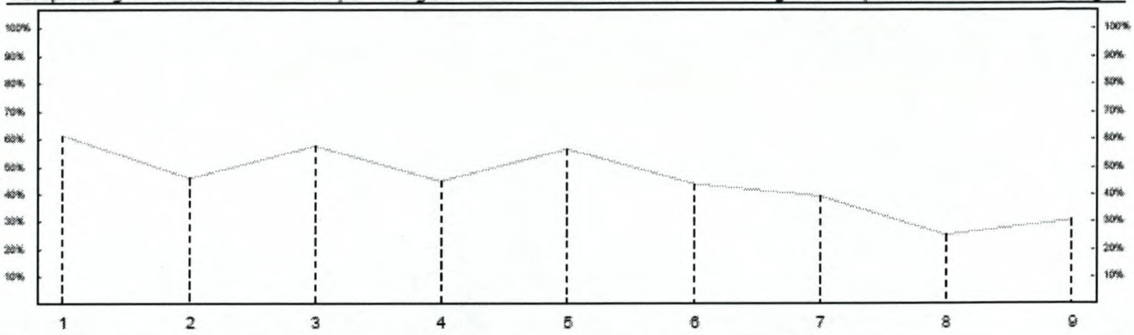
VRAAGONTLEDING / QUESTION ANALYSIS - NOVEMBER 2006

Verspreiding van Gemiddelde Punte per Vraag as Persentasies / Distribution of Average Marks per Question as Percentages



Gemiddelde Persentasie per Vraag / Average Percentage per Question

Question:	1	2	3	4	5	6	7	8	9	Paper
Max mark:	24	27	18	23	22	22	22	18	24	200
Average WCED:	61.0	45.5	57.3	44.5	56.1	43.4	38.9	24.9	30.2	43.8
No candidates:	5033	5042	5027	5016	4977	4913	4880	4767	4571	5062

SOURCE: WCED Examination Results 2006

Figure 8

It has been my experience that high school mathematics teachers blame the primary school teachers for the bad mathematics results. They argue that the primary school teachers allow the learners to progress despite the fact that the learners did not exhibit the desired mathematical competencies. Similarly university lecturers blame high school teachers for bad mathematics results at first year level. I think it is significant that each succeeding level of the educational system blames the previous level, since it alludes to the fact that students pass through the system with less than the expected level of mathematical knowledge and skills. This is especially so in the case of proof and proving as can be seen from the question analysis.

Kutzler **[58]** is of similar opinion as can be seen from the following quote: "In mathematics teaching at school we simply don't have enough time to wait until all students have completed all previous storeys. The curriculum forces the teacher to continue with the next topic, independent of the progress of individual students." The grade 12 mathematics results of the WCED (fig 4, 6 and 8) bears testimony to the fact that in South Africa this is also the case i.e. that some learners get passed on from grade to grade without acquiring the requisite skills and knowledge for the next grade. For example if a learner does not develop the competency to solve linear equations, then such a learner will struggle with the solving of quadratic equations, since steps in solving quadratic equations requires the solving of linear equations. This in turn will cause the learner to struggle with finding the x-intercepts of graphs of parabolas as this is dependent on solving quadratic equations, etc.

Add to the above argument the fact that in the majority of cases in mathematics assessment is utilized only for progression purposes and very rarely as a diagnostic tool (i.e. to determine and solve reasoning problems of students). The result is that students pass right through the educational system without their mathematical reasoning problems being addressed. I, for example, have discovered that some grade 12 learners still struggle

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with addition and subtraction of fractions. This is an example of the difficulty that learners develop with some aspects of numeracy at primary school level (as indicated in figures 1 and 2) and which is often also exhibited at high school level. This then alludes to the fact that numeracy problems such as addition and subtraction of fractions might have been uncovered by assessment at primary school, but has never been addressed.

The TIMSS **[47]** report of 2003 indicated that South African mathematics and science teachers are among the least qualified of the 50 countries that participated in the study. This can be a contributing factor of the low marks obtained by learners in questions on geometry in the WCED Examinations given in figures 4, 6 and 8.

1.3 The Aim of the study

The International Commission on Mathematical Instruction (ICMI) [5] **[48]** will have a study conference that will deal with the role of proof and proving in mathematical education in 2009 in Taipei in Taiwan. In their discussion document for this study, the organizers of the conference stated the following in connection with the role of logic in proof and proving. "The traditional assumption has always been that teaching students formal logic would easily translate into helping them understand the deductive structure of mathematics, and also to write proofs. But as research has shown this transfer doesn't happen automatically. It is still not clear what benefit, if any, may arise from teaching formal logic to students or to prospective teachers, in particular when mathematicians have been known to readily admit that they infrequently use formal logic in their research. Hence there is a need for more research that might provide support for the conclusion that the teaching of formal logic increases students' ability to prove or to understand proofs."

As a result of the above arguments and in an attempt to improve the ability of first year university mathematics students to read, understand and prove mathematical statements, we have embarked on teaching these students a course in mathematical logic in the hope that this will improve their mathematical proving abilities.
What we are not claiming is that we will get all the students to have the same ability i.e. that we will get all of them to become successful in proving mathematical statements. This is because not all students develop the same abilities while learning. Vygotsky puts it as follows: "When it was first shown that the capability of children with equal levels of mental development to learn under a teacher's guidance varied to a high degree, it became apparent that those children were not mentally the same age and that the subsequent course of their learning would obviously be different." We are also not claiming that the reasons advanced by us are the only reasons responsible for the bad mathematical results at school, we accept that there might be other factors involved that we have not addressed.

[5] The International Commission on Mathematical Instruction (ICMI) was established in 1908. The members of ICMI are neither individuals nor organizations, but countries. The ICMI currently consists of 72 member states. The focus of the ICMI is to enquire into mathematics teaching in countries world-wide.

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1.4 The Main Research Question

The main research question this study wants to answer is:

If first year university mathematics students are taught a course in elementary logic, would the acquired competency translate into increased ability to prove mathematical statements?

In order to answer this question we taught first year mathematics students at the University of the Western Cape a course in elementary logic. To determine their initial reasoning abilities we gave the students logical puzzles to solve without any prior teaching. We considered this as a pre-test. The students then received instruction on elementary logic after which they were given similar puzzles to solve which was considered as a post-test. This was done to ascertain if the students have acquired the ability to apply their knowledge. Subsequently the students received instruction on different types of proof and on conclusion of each type of proof were presented with a post-test. These pre- and post-tests were then analyzed statistically to establish if there was an improvement in proving ability.

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CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

The literature shows that a number of studies were done to determine the effects of a course in logic on student proving abilities. Some of these studies were done at high school whilst others were done in the first year at university level. The studies done with high school learners were done in the domain of Euclidean geometry. Four articles that investigated the effect of logic on proving at high school will be reviewed, whilst two articles will be reviewed that investigated the effect of logic on proving at tertiary level.

2.2 Effect of logic on proving at school level

The purpose of the study by G.W. Deer **[22]** of Florida State University was to determine what effects the teaching of an explicit unit in logic would have on students' ability to write proofs in high school geometry and in an unrelated mathematical system in which the students had no previous training. To accomplish this Deer divided a high school geometry class into two groups of 13 students each. One group was taught a unit in introductory logic including the basic inference patterns. The other group studied material on games, puzzles, codes and modular arithmetic during the time the first group was taught logic. At the time of the study the students had not been introduced to the concept of proof in geometry. Pre- and post- tests were administered to measure the students' understanding of the basic inference patterns in logic. The unit in logic, including the two tests, involved nine days of class work.

After these nine days the two groups were combined into one class and taught to prove theorems in geometry. Lessons on congruent angles, supplementary and complimentary angles, perpendicular lines, right angles and the intersection of lines was used to present the proof concept. The types of proof used were direct and indirect proof and the two-column format was used for all proofs. The axioms necessary for these lessons were introduced prior to the study. The unit in proof writing was done over nine days of class work.

Two tests were given upon completion of the unit in proof writing. One was a test of their ability to write proofs of the theorems in geometry. This test consisted of theorems the learners had not seen before, but were easily derived from the material studied. The other was a test of their ability to write proofs in a mathematical system in which they had no prior experience. For this test a set of 7 axioms was given and then using these axioms the students were asked to prove 5 theorems. Each test was done over an hour.

This study found that there was no significant difference in the performance of the two groups on each test. Hence the conclusion of the study was that a course in logic does not enhance the ability to write proofs in high school geometry and therefore logic does not improve the ability to do deductive reasoning.

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The study of John Lewis Platt **[70]** of Colorado state college was designed to evaluate the effect of the use of mathematical logic in high school geometry on: achievement of students in geometry; achievement of students in reasoning in geometry; critical thinking of students and attitude of students toward logic deduction and proof in mathematics. Two groups of six classes were involved in the study. One group was an experimental group of six classes who participated in a four week unit in mathematical logic and who applied the logic in the learning of deductive methods in geometry. The control group was taught the same course in geometry but without the study of mathematical logic. The conclusions of the study were as follows: mathematical logic is an appropriate area of study for both high and average achievers in high school geometry; there is no loss of achievement in geometry caused by devoting time from the traditional course in geometry to the teaching of mathematical logic; including mathematical logic in high school geometry does not result in a course which is significantly superior to the traditional course in its over-all effect upon student achievement in reasoning in geometry, critical thinking ability or attitude of students toward logic, deductive thinking and proofs in mathematics; including instruction in mathematical logic appears to produce a more effective treatment of high school geometry with high achieving students in its effect upon student achievements in reasoning in geometry.

The study conducted by David Mueller **[67]** was intended to determine the effects of the teaching of a logic unit on the proof writing abilities of high school geometry students in familiar and unfamiliar settings. The study was done at a Florida high school in the first half of the 1973 – 74 school year and involved four teachers, six classes and 146 students. Two classes were reserved for the higher ability students and the remainder of the classes had a regular mixture of ability levels. The two classes were treated differently. The higher ability group first studied the logic unit and then two geometry units, whilst the other group studied the logic unit between the two geometry units. The two geometry units consisted of chapters 1, 4 and 5 from the book on geometry by Moise and Downs (1964) which was the prescribed book. The logic unit was taken from a book written by Bastic in 1969. This book was unusual in three main ways, namely: it presented the logic informally rather than in formal syllogisms; it contained material in the form of Lewis Carrol-type puzzles to assist in learning to draw conclusions and to determine which step comes next, which are important items in writing proofs; it contained material to specifically assist in using the interpreting axioms; it made liberal use of nonsense content in the examples.

To verify that the students learned the logic material a test was given before and after the logic unit. A test of ability to interpret and use axioms was given before, during and after the logic unit to provide a check on the effectiveness of materials in the unit designed to increase this ability. An attitude scale was utilized to evaluate changes in attitude towards mathematics. Tests of proof writing ability in both familiar (using geometry content just studied) and unfamiliar settings were administered after each of the two geometry units in order to test the effects of the logic unit on proof-writing ability.

The main research questions were concerned about the effect on proof-writing skills. The finding was that the logic unit was of little significant help in writing proofs of theorems

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in either the familiar or unfamiliar settings. Another finding was that the logic unit appeared to have some initial negative effect, but some beneficial effect later on. The results however were mixed for the various classes and showed no completely consistent pattern. The minor research questions concerned the change in attitudes and ability to interpret and use axioms. One of the findings was that there was some decline in attitude toward mathematics, but it could not be determined whether it was due to the logic unit or at least partially to other causes. The other finding was that the last part of the logic unit very significantly aided in improvement of the ability to interpret and use axioms as it was intended to .

The general conclusion was that the logic unit was only partially successful in answering the research questions. The researcher therefore recommended that a more effective approach would be an integrated approach. He suggested that this should include presentation of the key logic techniques and the use of explicit reminders that these techniques are embodied in the examples.

Epp **[34]** in her summary of the Texas Pre-freshman Engineering program (Tex Prep) of Berriozabal **[5]** states that the study reports considerable long-term impact of the study of logic on the students in the program. This was a comprehensive program intended to identify middle and high school students with the potential and interest in becoming engineers and scientists and to guide them toward acquiring knowledge and skills required for success in their professional aspirations. Prep was an academically intense eight week summer program which stresses the development of abstract reasoning skills, problem solving skills and their application. The report indicates that since 1979, 8067 students have successfully completed at least one summer of Tex Prep. The program was run over 3 years and in the first year logic and its application to mathematics was done. The research was done by means of questionnaires.

2.3 Effect of logic on proving at tertiary level

The aim of the study done by R.L. Walter **[90]** of Florida state university was to determine the effect of knowledge of logic in proving mathematical theorems in the context of mathematical induction. His study was aimed at determining if knowledge of logic would cause people to perform better on tests on the principle of mathematical induction than people that have no knowledge of logic. In order to determine this, instructional material in logic and the principle of mathematical induction were presented to the students.

The logic unit was not programmed, whereas both of the mathematical induction units traditional and experimental were programmed. The experiment was run twice, using pre-calculus college students first and college calculus students the second time. The test that was administered consisted of two parts, namely, multiple choice and proof. In both cases the results were in favour of the research hypothesis, however only the variable proof with pre-calculus students was significant at the .05 probability level. The results of the calculus students on the other hand were far from significant with probability values greater than .50.

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Using the ACT[6] English, mathematics and composite scores to define ability levels, it was apparent that the students from the calculus classes had a higher ability level than those students in the pre-calculus classes. In the pre-calculus class students had a wider range of abilities than those from the calculus classes. The investigator therefore concluded that the effect due to different treatments was not as evident in the calculus students because of the high ability level present. In other words the calculus students did well in spite of the treatment. In view of the results obtained with both the pre-calculus and the calculus students the researcher concluded that the research hypothesis was not without merit since in both cases the results favoured the research hypothesis. He therefore suggested further research into the effect of the knowledge of logic on different instructional strategies in the principle of mathematical induction.

Cheng et al **[14]** are of the opinion that two views have dominated theories of deductive thinking. One is the view that people reason using syntactic, domain independent rules of logic and the other is the view that people use domain-specific knowledge. In contrast to the above two views Cheng et al are of the opinion that people often reason using pragmatic reasoning schemas. These schemas are clusters of rules that are highly generalized and abstracted but nonetheless defined with respect to classes of goals and types of relationships. The researchers therefore designed three experiments, using college students, to examine the processes involved in deductive reasoning.

In experiment 1 it was investigated how people interpret and reason about a type of logical statement, called the conditional. The researchers randomly assigned 80 students (in equal groups) of the University of Michigan to each of four training groups namely: rule training; examples training; rule plus examples training and; no training. None of the chosen students had previously received any formal training in logic.

The training materials and methods were as follows: for rule training the students received a seven-page booklet containing an exposition on conditional statements followed by an inference exercise. The exposition consisted of an explanation of the equivalence between a conditional statement and its contra-positive, as well as an explanation of the two common fallacies of affirming the consequent and denying the antecedent. The contra-positive was explained in part by the use of a truth table and in part by Euler diagrams that used concentric circles to show the relations between a conditional statement and its contra-positive and in part by an illustrative conditional statement. Students were given immediate feedback on correctness of the inference exercises as well as a brief explanation of the correct answer.

With the examples training students were requested to solve two selection problems[7]. Students were given immediate feedback about their performance. The Rule plus Example training consisted of the materials for the rule condition followed by those for

[6] American College Testing (ACT) – Is a national USA college admission and placement examination. It consists of four tests namely English, Mathematics, Reading and Science. The score range for each of the four tests is 1 – 36. The composite score as reported by ACT is the average of the four test scores earned during a single test administration rounded to the nearest whole number.

[7] Wason's selection task was used by the researchers.

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the examples condition. The only further addition was that for these students the explanation of the correct answer for each example was given in terms of the abstract rules they had just learned. The students were also given eight selection problems as a test. This test consisted of two of each of three types of problems involving a bi-conditional rule.

The finding of experiment 1 was that training in standard logic when coupled with training on examples of selection problems leads to improved performance on subsequent selection problems. In contrast, training on rules of logic without such examples failed to significantly improve performance. The researcher therefore concluded that this is consistent with their view that the material conditional is not part of people's intuitive reasoning abilities and that because of this they lack any ability to put abstract rule training to use.

With experiment 2 the researchers examined the impact of a one-semester undergraduate course in standard logic. The students of two introductory logic classes were used as subjects. One class was held at the Ann Arbor campus of the University of Michigan and one at the branch campus at Dearborn. Both classes received 40 hours of lectures and covered topics in propositional logic, including modus ponens, modus tollens, affirming the consequent and denying the antecedent, and the distinction between the conditional and the bi-conditional. The textbook used in one class was Elementary logic by Simco and James and in the other it was Introduction to Logic by Copi. The course utilized meaningful conditional sentences to illustrate the inference rules and fallacies. The emphasis however was placed on formal logical analyses i.e. truth-table analyses and construction of proofs.

A pre-test and post-test were administered to each class. The pre-test was given in the first week of class before any discussion had taken place. The post-test was given in the final week of the semester. The eight selection problems used in experiment 1 were divided into two sets of four and were given in a booklet to students to complete during lectures. No feedback was provided until after completion of the post-test. Only data from students who completed both the pre-test and the post-test were analyzed, so that the effect of logic training could be treated as a within-subject variable. A total of 53 students completed the study.

The findings was that abstract training in the logic of conditional statements does not have much effect on the way people reason about problems that could potentially be solved by its use. On the other hand problems that can be interpreted in terms of pragmatic reasoning schemas were solved by a large proportion of the students. The researchers therefore were of the opinion that it might be possible to improve people's deductive reasoning by training them on pragmatic reasoning schemas.

In experiment 3 the researchers wanted to test the above possibility. For this experiment 72 University of Michigan undergraduates were selected. The students were randomly assigned in equal numbers, to one of three groups. One group was assigned a control condition in which no training was given, whereas the second group was assigned to an

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obligation-training condition and the third to a contingency-training condition. Students that were assigned to the two training conditions were given the appropriate training materials to read for 10 minutes and were then asked to complete the test problems. The control subjects, i.e. the students that received no training, were simply given the test problems and asked to complete them.

The obligation training material consisted of a two-page booklet that contained details on the nature of obligations, the procedures necessary for checking if a violation of the obligation has occurred and also an example of an obligation statement presented for assessing obligations were described in terms of four rules, one for each of the four possible situations that might arise, and were represented by *p*, *not p*, *q* and *not q*.

The contingency training material was similar to the obligation-training material except that the checking procedures were described in terms of assessment of "contingencies" rather than "obligations". The test- problem booklet consisted of eight selection problems. Four of these problems were easily discernible as obligation situations and the other four were arbitrary. The finding of this experiment was that it was effective for at least the arbitrary problems.

Based on the above three experiments the researchers concluded that people typically reason using abstract knowledge structures organized pragmatically, rather than in terms of purely syntactic rules of the sort that comprise standard logic. They found that students reasoned in closer accord with standard logic when thinking about problems intended to evoke permission or obligation schemas than when thinking about purely arbitrary relations.

The training results showed that an entire course in standard logic had no effect on the avoidance of any error except a slight reduction in the fallacy of affirming the consequent. A brief training session in formal logic of a type that was proved to produce substantial effects on people's ability to reason using the law of large numbers, had no significant effect on students' ability to use modus ponens or modus tollens or to avoid the errors of affirming the consequent or denying the antecedent. The researchers however was of the opinion that rule training was not useless, since when it was combined with examples training, students were able to make substantial use of it. They furthermore concluded, based on results that since modus ponens are not a universal rule of natural logic that therefore it is highly unlikely that any formal deductive rule is general across the adult population.

The researchers stress that one of the educational implications of the study is that deductive reasoning is not likely to be improved by training on standard logic. They however also state the following: " …if logic instructors wish to influence their students' inferential behaviour in the face of novel problems, they must do much more than they currently do to show how to apply logical rules to concrete problems." So it seems that by providing guidance as to where and how logic should be applied might subvert this educational implication.

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2.4 Literature review summary

All three articles (Deer **[22]**, Platt **[70]**, Mueller **[67]**) dealing with high school geometry concluded that logic does not help to improve proof-writing of high school students. Platt however also conclude that logic does help teachers to deal more effectively with geometry and that it is an appropriate area of study for both high and average achievers. Mueller found that logic did help students to improve in their ability to interpret and use axioms in proving. He recommended an integrated approach to the teaching of logic, that include the presentation of logic techniques and the use of explicit reminders as to where and when to employ logic.

Cheng et al **[14]** concluded in their study with university students that training on logic does not improve deductive reasoning and hence proving abilities of students. They do however show that in some instances training on pragmatic reasoning schemes helped to improve the deductive reasoning of people. Another suggestion is that instructors at tertiary level should show their students how to apply logical rules to concrete problems in order to enhance their inferential behaviour.

On the contrary the study done by Walter **[90]** with college students showed that logic does help students to improve their proving abilities. The study however only researched the effect of logic on proof in the context of mathematical induction. This study therefore paved the way for research into the effects of knowledge of logic on proving abilities using other methods of proof used at tertiary institutions. Hence our study researched the effect of knowledge of logic on direct proof, contradiction, contraposition and mathematical induction in the context of number theory and set theory.

The study by Mueller **[67]** made use of Lewis-Carol-type logic puzzles to assist the students to learn to draw conclusions and to determine which step comes next in writing proofs. Our study also made liberal use of this informal way of introducing proof techniques to students and one of the aims of the study was to determine if this method was effective in helping students to improve their deductive abilities.

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CHAPTER 3

THEORETICAL FRAMEWORK

3.1 Introduction

To put proof and proving in perspective in the current educational and research environment, I am of the opinion that one should start with some elementary questions like: What is proof in mathematics? Is proof important in mathematics and if so why?

3.2 Is proof important in mathematics and if so why?

Let us start with the second question that refers to the importance of proof in mathematics since it seems to be less contentious. There seems to be general consensus that proof plays a central role in mathematics. Jones and Rodd **[50]** of the British Society for Research into Learning Mathematics state: "Proof and proving are of course, central to all mathematics". Volmink **[88]** advances the following argument as to the importance of proof: "It may be asserted that proof is the essence of mathematics". Knuth **[53]** is also convinced that proof is very important in mathematics as the following quote shows: "Many consider proof to be central to the discipline of mathematics and the practice of mathematicians." Herbst **[44]** answers the question why proof is important in mathematics education as follows: "Proof is essential in mathematics education not only as a valuable process for students to engage in (such as developing their capacity for mathematical reasoning), but more importantly as a necessary aspect of knowledge construction." Stylianides and Stylianides **[80]** advances the following reasons as to the importance of proof: "…proof is fundamental to doing mathematics – it is the basis of mathematical understanding and is essential for developing, establishing, and communicating mathematical knowledge. Secondly, students' proficiency in proof can improve their mathematical proficiency more broadly because proof is involved in all situations where conclusions are to be reached and decisions to be made." Also proof is seen by some to go hand–in-hand with the deductive reasoning ability of students as can be seen from the following quote in Stylianides and Stylianides: "In addition, from a psychological standpoint, the development of students' ability for deductive reasoning has been found to go along with the development of their ability for proof."

The question as to what proof is, is not as straightforward as there seems to be conflicting views as to what proof is. The following quote from Kleiner **[52]** based on Gödel's incompleteness theorems make this abundantly clear: "Gödel's work demands the surprising and for many discomforting conclusion that there can be no definitive answer to the question "what is proof?". Reid **[71]** is convinced that there is no consensus among mathematics researchers and teachers as to what proof is and he therefore states: "But if we can acknowledge that there is a problem and discuss the characteristics of proof, we may be able to come to, if not agreement, then at least agreement on how we differ."

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3.3 Origins of proof

In order to explain and understand the concept proof, I believe one needs to start at the origins of proof. There is general consensus among the mathematical history pundits that the first people that introduced the concept of mathematical proof were the ancient Greeks (Kleiner **[52]**, Kutzler **[58]**, Barker **[3]**, Stillwell **[79]**). There are however other nations that contributed enormously to the mathematical knowledge, concerning proof that is available today. Two of these nations are the Egyptians and Babylonians. Their mathematics however lacked the concept of proof (Kleiner **[52]** ; Barker **[3]**; Kutzler**[58]**) and was based on inductive reasoning. But what is inductive reasoning?

According to Ayalon and Even **[2]** there are various sorts of thinking and reasoning including creation, induction, plausible inference and deduction. For our purposes we are going to elucidate only inductive and deductive reasoning. Ayalon and Even defines inductive reasoning as follows: "…developing hypotheses based on empirical observations to describe "truths" or "facts" about our world." They define deductive reasoning as follows: " Deductive reasoning is unique in that it is the process of inferring conclusions from known information (called premises) based on formal logic rules, where conclusions are necessarily derived from the given information and there is no need to validate them by experiments." Stylianides and Stylianides **[80]** concur with this definition of deductive reasoning. They see it as the general form of reasoning associated with logically necessary inferences based on given sets of premises.

One way of putting the above methods of reasoning into perspective is to look at Buchberger's Creativity spiral as explained by Kutzler **[58]**. Buchberger's spiral provides the different aspects that are involved in the discovery of mathematical knowledge. In the spiral there are three phases namely experimentation, exactification and application. Kutzler explains that during the phase of experimentation one applies known algorithms to generate examples then obtains conjectures through observation. During the phase of exactification conjectures are turned into theorems through the method of proving, then algorithmically useful knowledge is implemented as algorithms. During the phase of application one applies algorithms to real or fictitious data. Using the above explanation Kutzler asserts that the Egyptians and other ancient civilizations applied only the phases of experimentation and application in their construction of mathematical knowledge. So they only used *inductive reasoning*. He argues that in about 500 B.C. the Greeks took the Egyptian mathematics and applied to it the *deductive method of reasoning* i.e. they added the phase of exactification. He maintains that from then on mathematics comprised of all three phases and that mathematics was thus established as the deductive science of today.

However from about 1950 on the French mathematician Dieudonne and his colleagues (known as the Bourbaki group) developed the system of "definition-theorem-proof-corollary-…" The Bourbaki system therefore did not include the phase of experimentation and consists only of the phases of exactification and application. This Bourbaki system has now become part of the modern process of teaching and learning. Thus it has become customary to teach mathematics by deductively presenting

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mathematical knowledge and then asking students to learn it and then use it to solve homework and examination problems.

Since the deductive method is prevalent in mathematics, I think it is imperative that it is subjected to closer scrutiny. The obvious place to start would be with geometry, since the Greeks that started the deductive method is synonymous with geometry. There were a number of Greek thinkers that gave attention to geometry, but it was Euclid that presented in systematic form, all the main geometrical discoveries of his predecessors in his classic book "The Elements". This book played an influential role in Western thought as can be seen from the following quote from Barker **[3]**: " Through ancient times, through the medieval era, and in the modern period right up into the nineteenth century, Euclid's Elements served not only as the textbook of geometry but also as a model of what scientific thinking should be." I am of the opinion that an important question now is: What are the salient features of Euclid's procedures that distinguishes it from the inductive method?

The nations that applied the inductive method were satisfied with showing that a principle holds for a number of particular cases by means of experimentation and observation. This however did not show that it is true in general but only for the particular cases studied. Conversely a distinctive feature of Euclid's Elements is that he formulates his geometrical laws in universal form i.e. he proves that his laws are true in ALL cases. Barker **[3]** states of Euclid in this regard: "He never concerns himself with actual experiments or observations like that. Instead his proofs are deductive proofs by means of which he seeks to establish his conclusions with the rigor of absolute logical necessity." But what is this proof that was now being introduced as a mathematical procedure?

Barker **[3]** defines proof in the classic sense as follows: "A proof is a chain of reasoning that succeeds in establishing a conclusion by showing that it follows logically from premises that already are known to be true. We cannot have a proof unless we can start with one or more already known premises which serve as a basis upon which the proof is to rest." What Euclid did in order to use known premises to start his proofs from was to divide the geometrical laws into two groups. The postulates are a small group of laws which Euclid thought are self-evident and therefore need no proving, but which he adopted as basic premises. Examples of these are laws about lines, angles and figures, laws which Euclid regarded as true and which he employed for the proof of other laws. Conversely Euclid assumed that there were an infinitely large group of *other* geometrical laws which can be proved using these postulates. This second group of laws he named theorems or propositions.

Euclid did not only deal with geometrical laws, but also with principles that deal with equality of magnitude. These five other principles were called axioms. A well known example of one of these axioms is: "the whole is greater than the part." In the modern era the distinction between axioms and postulates has become blurred and today the words axioms and postulates are used interchangeably. All of the above discussion about Euclid's methodology is succinctly put by Barker as follows: "The postulates, axioms

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and definitions supply the starting point for Euclid's proofs. His aim is to prove all his other geometrical principles, first those of plane geometry and then later on those of solid geometry, by showing that they follow necessarily from the basic assumptions". So in the Euclidean system the axioms were regarded as self-evident truths and therefore there was no need to prove them. However with the development of the non-Euclidean geometries of Lobachevsky, Bolayi and Riemann it became clear that not all propositions within a formal system can be proved without getting circularity and that consequently certain propositions had to be accepted as starting points. De Villiers **[29]** therefore argues as follows: "Whereas many had previously believed that axioms were "self-evident truths", they now realized that they were simply "necessary starting points" for mathematical systems."

The above then is how proof in mathematics was started. Since then however the concept of proof has been vigorously debated. In the 19th century there were frequent disagreements among mathematicians concerning what they thought what the foundations of mathematics was. In the early 20th century this disaffection among mathematicians gave rise to three schools of mathematical thought. These schools of thought gave rise to the philosophies of logicism, formalism and intuitionism.
Kleiner **[52]** and Hanna **[42]** are convinced that this was the first formal expression by mathematicians of what mathematics is about and in particular, of what proof in mathematics is about.

The logicists advocated that mathematics is part of logic. To them mathematical concepts are expressible in terms of logical concepts and mathematical theorems (proofs) are tautologies i.e. are true by virtue of their form rather than content
(Kleiner **[52]**, Hanna **[42]**)

The formalists viewed mathematics as a study of axiomatic systems. According to this philosophy both the primitive (undefined) terms and the axioms are considered to be strings of symbols to which no meaning is to be attached. These primitive terms and axioms are to be manipulated according to established rules of inference to obtain the theorems of the system. (Kleiner **[52]**)

The intuitionists on the other hand claimed that no formal analysis of axiomatic systems is necessary. They believed that the mathematician's intuition, beginning with that of number, should guide him to avoid contradictions and that he must pay special attention to definitions and methods of proof. For the intuitionists mathematics and mathematical language are two separate entities with mathematical activity essentially a languageless activity of the mind. They argue that their methods of proof must be constructive and finite in nature. (Kleiner; Hanna)

3.4 Functions of proof

According to Griffiths **[41]** the traditional view of proof is as follows: "A mathematical proof is a formal and logical line of reasoning that begins with a set of axioms and moves through logical steps to a conclusion." In this view of proof there is an emphasis on form

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and systematization i.e. starting with axioms and then by reasoning deductively coming to a conclusion. This view however does not enjoy complete support among mathematicians and mathematics educators today. This lack of support in some quarters is confirmed by Hanna **[43]** in the following quote: "In the last two decades several mathematicians and mathematics educators have challenged the tenet that the most significant aspect of mathematics is reasoning by deduction culminating in formal proofs…Mathematicians agree, furthermore that when a proof is valid by virtue of its form only, without regard to its content, it is likely to add very little to an understanding of its subject and ironically may not even be very convincing." As a result of this disagreement with the traditional view of proof, a more recent view of proof is that proof is an argument needed to validate a statement, an argument that may assume several different forms as long as it is convincing (Hanna **[43]**). This alternative view of proof, that does not only depend on logical structure, but also on how convincing it is, enjoys different interpretations as to who must be convinced. Weber **[91]** supplies some of these alternative views. He argues that some of the mathematical fraternity (Maso, Burton and Stacey) see proof as an argument that must convince an enemy. Others such as Davis and Hersch see proof as an argument that must convince a mathematician who knows the subject, while Volmink **[88]** and Epp **[34]** sees proof as an argument that must convince a reasonable skeptic. Balacheff and Manin focused on the social and contextual nature of proof. Balacheff define proof as an explanation accepted by a given community at a given time, while Manin argues that an argument becomes a proof only after the social act of accepting the argument as a proof.

It would seem, based on the above arguments about proof, the only function of proof is that of convincing. There are some however that assign functions other than convincing to proof. The following list of functions of proof is provided by Hanna **[43]**: verification i.e. concerned with the truth of a statement; explanation i.e. providing insight into why it is true; systematization i.e. the organization of various results into a deductive system of axioms, major concepts and theorems; discovery i.e. the discovery of new results; communication i.e. the transmission of mathematical knowledge; construction of an empirical theory; exploration of the meaning of a definition or the consequences of an assumption; incorporation of a well-known fact into a new framework and thus viewing it from a fresh perspective. The development of mathematical intuition is seen as another function of proof by Pinto and Tall. Yackel and Cobb are convinced that by teaching students how to prove they will develop the ability to independently construct and validate new mathematical knowledge. They are therefore of the opinion that teaching students how to prove will cause them to become autonomous in mathematics.

Reid **[71]** argues that there is a dimension of proof that is different from the dimensions of concept and purpose that we need to consider. He identifies this as the reasoning involved in proving. He argues that there are three prevalent views on the kind of reasoning involved in proving. These are that either there are several distinct kinds of reasoning involved in proving or several kinds of reasoning in combination are involved in proving or deductive reasoning alone constitutes proving. He argues further that those with a traditional concept of proof would be more likely to see proof as purely deductive,

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while those with a quasi-empirical concept of proof might be more inclined to view proving as a combination of kinds of reasoning.

3.5 Proving in South African schools

Since our study targeted first year students it is imperative that we determine what their previous experience with proof and proving has been at school level.

In South Africa a new curriculum known as the National Curriculum Statement (NCS) was implemented in grade 10 in 2006. Our target group of students however still dealt with the old curriculum so we are going to look at proof and proving in the old curriculum. In this curriculum Euclidean geometry usually was the topic used to introduce formal proof to learners. In South Africa the two column-proof format is used extensively in Euclidean geometry. But what is a two-column proof and why is it used? Herbst **[45]** argues that Arthur Schultze and Frank Sevenoak were the first to use the two-column format for writing out proofs in geometry. In this format statement and reasons were written next to each other divided by a vertical line. Herbst quotes Schultze and Sevenoak to elucidate the use of the two-column proof: "Every proof consists of a number of statements each of which is supported by a definite reason. The only admissible reasons are: a previously provided proposition; an axiom; a definition; or the hypothesis." The justification provided for giving reasons for each statement is that the reasons given by the student will enlighten the teacher as to the understanding of the student of the deductive argument involved in the composition of the proof.

The geometry courses of the USA were composed of two different kinds of propositions namely Fundamental propositions and exercises. The Fundamental propositions are described as the minimum which all pupils should know. These Fundamentals would serve to develop the subject of studies and to exemplify what it meant to prove a proposition. On the other hand the Fundamentals themselves would be the information that students would use as they did proof exercises. Herbst **[45]** maintains that these exercises were meant to stimulate student reasoning and to practice what had already been learned. South Africa has a similar system. In South Africa the Fundamental propositions are the prescribed theorems and axioms. These prescribed theorems and axioms formed the underlying knowledge that each learner has to know in order to do proof exercises. It was expected of the teacher to present proofs of these theorems in the statement and reason format to learners. Some of the reasons advanced for this format, was so that students could understand the deductive reasoning involved and also to serve as an example of how proving should be done. De Villiers **[28]** writes about this as follows: "The rationale for including formal geometry in the school curriculum is two-fold: it is seen as a vehicle for teaching and learning deductive thinking "proof" and also as a first encounter with a formal axiomatic system." It is therefore not surprising then that some teachers and learners at school are under the impression that this is the only kind of proof there is, and that all proofs should be done in this way as the two-column proof form the bulk of their proving experiences at school level. In fact research done by us with 50 teachers in one of the EMDC's has shown that only two out of the 50 teachers

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knew of other types of proof.[8] Weber **[91]** writes the following in this regard: "One reason that university students find proof so difficult is that their experience with constructing proofs is typically limited to high school geometry." Stylianides et al **[80]** is of similar opinion as can be seen from the following quote: "… several researchers have identified students' abrupt introduction to proof in high school as a possible explanation for the many difficulties that secondary school and university students face with proof, thereby proposing that students engage with proof in a coherent and systematic way throughout their schooling."

In 1996 De Villiers **[27]** claimed that it is a well known fact that learners in grade 12 in South Africa perform much worse in Euclidean geometry than in algebra. Eleven years later in 2007 it was still the case as we showed in the introduction in the tables of grade 12 results. Problems with proving are not restricted to grade 12, but become apparent in grade 8 and continue right through the system. Internationally Euclidean geometry has also been a thorn in the side of learners. For example De Villiers **[27]** writes the following of the Russians: " In the late sixties Russian researchers undertook a comprehensive analysis of both the intuitive and systematization phases in order to try and find an answer to the disturbing question of why pupils who were making good progress in other school subjects, showed little progress in geometry." The question is why do learners struggle so much with Euclidean geometry?

The Van Hiele theory provides some answers to the above question. Pierre van Hiele and Dina van Hiele-Geldof were a husband-and-wife team of Dutch researchers who noticed that their students had difficulties in learning geometry. Their observations led them to develop a theory involving levels of thinking in geometry that students pass through as they progress from merely recognizing a figure to being able to write a formal geometric proof. According to de Villiers **[27]** the Van Hieles attributed the dismal performance by students in geometry to the fact that the curriculum was presented at a higher level than those of the students. As a result the students could not understand the teacher and the teacher could not understand why the students could not understand.

Young **[93]** argues that according to Piaget in order for students to gain the ability to construct a proof, they must progress through certain stages of learning. At stage one a student is non-reflective, unsystematic and illogical. At this level students will explore randomly without a plan and will be unable to generalize from one example to the next. At stage two students will begin to establish relationships, anticipate results and think logically about premises they believe in. Students at this level are still not ready to conquer the concept of proof but are beginning, on their own to use some informal reasoning to justify conclusions. At stage 3 (known as the formal operational stage) students are capable of formal deductive reasoning and can operate within a mathematical system. At this stage students begin to see that because a statement is always true implies that it necessarily must be true. Although the age at which deductive reasoning starts is disputed educational research has shown that different forms of deductive reasoning can start as early as the elementary grades as can be seen from the following quote from

[8] This research was done by means of a questionnaire that was presented to teachers in the southern EMDC (Education management District Centre) in Cape Town, South Africa - See appendix J.

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Stylianides et al **[80]**: "Even though the findings of existing psychological research do not specify exact ages at which students master different forms of deductive reasoning, all forms of deductive reasoning we reviewed begin to emerge in the early elementary grades." However since Piaget's theory deal with understanding in general while the Van Hiele theory deals specifically with geometry we are going to take a closer look at the Van Hiele theory.

Four important characteristics of the Van Hiele theory as summarized by Usiskin **[87]** are as follows: Learners progress through the thought levels in a fixed order. In other words the learner has to pass through the levels sequentially i.e. the student must first pass through level one before he/ she can get to level two, etc. At each level of thought that which was intrinsic in the preceding level becomes extrinsic in the current level. Each level has its own distinct linguistic symbols and own network of relationships connecting those symbols. Two persons who reason at different levels cannot understand each other.
The Van Hiele theory distinguishes between five different levels of thought. These are given by Mason **[62]** as follows:
level 1 (visualization): student recognizes figures by appearance alone, often by comparing them to a known prototype. At this level students make decisions based on perceptions not reasoning.
Level 2 (analysis): students see figures as collections of properties. They can recognize and name properties of geometric figures, but they do not see relationships between these properties.
Level 3 (abstraction): students perceive relationships between properties and between figures. At this level, students can create meaningful definitions and give informal arguments to justify their reasoning. The role and significance of formal deduction, however, is not understood and hence proof is not understood.
Level 4 (deduction): students can construct proofs, understand the role of axioms and definitions and know the meaning of necessary and sufficient conditions. At this level students should be able to construct proofs such as those typically found in a high school geometry class.
Level 5 (rigor): students at this level understand the formal aspects of deduction, such as establishing and comparing mathematical systems. Students at this level can understand the use of indirect proof and proof by contrapositive and can understand non- Euclidean systems.

Since the majority of students in our study were Black and Coloured from previously disadvantaged groups in South Africa it is imperative that we determine what research says about their proving abilities. It is important to note that the majority of these students study mathematics in their second language. Research done by de Villiers and
Njisane **[26]** in 1987 has shown that about 45% of black learners in grade 12 in Kwa-Zulu Natal had only mastered Van Hiele level 2 or lower, whereas the examination required mastery at level 3 and beyond. De Villiers quotes research done by Malan (1986), Smith and De Villiers (1990) and Govender (1995) that shows similar results. He argues that the transition from level 1 to level 2 is particularly problematic for learners that have English as a second language since it involves the acquisition of the technical terminology by which the properties of figures need to be described and explored.

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De Villiers **[29]** is of the opinion that improvements in school geometry results are dependent on major revisions in primary school geometry. He uses the example of a Russian experimental curriculum based on the Van Hiele theory to prove his point.

In the late sixties the Russians wanted to determine why students that showed progress in other subjects performed worse in geometry. They came to the conclusion that the main reason for this state of affairs was insufficient attention to geometry in the primary school. They discovered that in the first five grades learners dealt mainly with Van Hiele level 1 activities, while from grade 6 learners suddenly had to deal with activities at a level 3 understanding. The Russians subsequently developed a very successful experimental geometry curriculum based on the Van Hiele theory. In this experimental curriculum they developed the learners' understanding sequentially and hierarchically (based on the Van Hiele levels) from grade 1. This had as a result that the average grade 8 of the experimental curriculum showed the same or better geometric understanding than their grade 11 and 12 counterparts in the old curriculum.

In South Africa until very recently the version of Euclidean geometry that was followed in high school was one in which congruency and similarity is a central theme. With this congruency geometry an axiomatic-deductive system of proving was utilized.
De Villiers **[29]** argues that an axiomatic-deductive approach to teaching is used when an unfamiliar topic is presented to students by means of the initial introduction of the axioms and definitions of that topic and logically deriving the other statements (theorems) and properties from them. This approach however has its limitation and a number of authors have critiqued the deductive presentation of mathematics. Some like Kline believe that students may develop feelings of inferiority since the deductive presentation of mathematics might lead the students to believe that mathematics is created by geniuses who start with axioms and reason directly and flawlessly to theorems. De Villiers lists the following as added criticisms of the axiomatic deductive approach to teaching. Learners cannot interpret work presented in this manner as axiomatic structures and deductive proof belong to Van Hiele level 4 and most learners enter high school only at the first or second Van Hiele level. Also no provision is made in such an approach to help learners progress towards the necessary or required levels. The teaching of axiomatic structures as finished products lead to the rote learning and memorization of axioms, theorems and proofs with little or no understanding of their meanings. The students do not become skilful in the application of creative mathematical processes like abstraction, generalization, defining, drawing of analogies, systematization, construction of proofs, etc. since only the end products of these processes are given directly to them.

Since teachers at school play a crucial role in learners' understanding of proof we have to determine what their perceptions of proof are. During a 1984 country-wide survey at 11 South African universities De Villiers **[26]** found that nearly half of the students in mathematics education had a traditional view of proof i.e. proving based on deduction. These students therefore believed in an axiomatic deductive approach to teaching proof.

It would seem then based on the above arguments that problems that learners have with geometry in the lower grades is carried right through the system and contributes

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enormously to the problems learners has with proving. Thus to change this state of affairs one has to start with the lower grades, making certain that they go through the Van Hiele levels hierarchically and sequentially. This should not only be the case for geometry, but should include proof in general, where learners should be given experience in proof not only where proof is done to verify, but also to explain, to systematize, to discover, to communicate, etc. This should be done right through all the grades so that learners will eventually become adept at proving. Several researchers agree with this as can be seen from the following quote from Stylianides et al **[80]**: "...several researchers have identified students' abrupt introduction to proof in high school as a possible explanation for the many difficulties that secondary school and university students face with proof, thereby proposing that students engage with proof in a coherent and systematic way throughout their schooling."

3.6 The distinction between a priori and a posteriori knowledge

Barker **[3]** a mathematical philosopher asserts that a distinction that have long been a topic of discussion in the philosophy of mathematics has been the distinction between a priori and a posteriori (or empirical) knowledge. He argues that this distinction is regarded as fundamental in philosophy of mathematics and I would argue that this is also very important in the teaching and learning of proof in mathematics as this will determine how one views the attainment of knowledge in proving.

In order to explain empirical knowledge he uses the example of someone that knows ravens are black. Now in order for the person to know this the person must understand what is meant by black and what is meant by raven. So the person must have seen ravens or have heard reports of ravens. In other words the person must have experienced this by means of their senses which include seeing, hearing, feeling, smelling or tasting. Hence the point is that only sensory observations can provide the kind of justification needed to entitle a person to say that he/ she knows facts like these. Therefore a posteriori knowledge can be defined as that knowledge that requires justification from experience.

To explain a priori let us examine an example of number theory. If a number is an even number then it is divisible by two; no prime number is divisible by two; therefore no prime number is even. To come to the conclusion that prime numbers is not even one does not need experience, but instead used deduction based on the first two premises. So this is purely a cognitive exercise and knowledge gained in this way is a priori. A priori knowledge therefore does not depend on experience and hence can be defined as knowledge that does not need to be justified by experience.

The knowledge of this distinction between a priori and a posteriori is important since it allows us to see that subjects like physics and biology are concerned with empirical knowledge since they rely on observations to reach conclusions. In contrast mathematical logic is concerned only with a priori knowledge since it only seeks knowledge of the rules governing the validity of an argument and therefore need not rely on observations to reach conclusions in its arguments. I would therefore argue that since logic and the majority of proofs are only a priori this is why students find it so difficult to apply and

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teachers find it difficult to teach since all the reasoning takes place cognitively and therefore one does not know where the reasoning problems are since the reasoning process is not readily observable.

3.7 Cognitive processes involved in learning proof

Argument form plays a crucial role in deductive logic. Epp **[34]** has the following to say in this regard: "The central concept of deductive logic is the concept of argument form." But what constitutes an argument? An argument can be seen as a sequence of statements aimed at demonstrating the truth of an assertion. Stated differently it means that one can only be confident in the conclusion that you draw from an argument if the statements composing it either are acceptable on their own merits or follow from preceding statements. The following statement from Epp stresses the importance of logic in mathematical reasoning: "Logical analysis won't help you determine the intrinsic merit of an argument's content, but will help you analyze an argument's form to determine whether the truth of the conclusion follows necessarily from the truth of the preceding statements. For this reason logic is sometimes defined as the science of necessary inference or the science of reasoning." To get a better understanding of argument forms and deductive reasoning I think it is imperative that we examine the cognitive processes and mental structures involved in this kind of reasoning.

McNally **[66]** in his study on Piaget and education use the example of a small child that sees sheep for the first time. The child exclaims: "Look at the puppy dogs." The reason McNally advanced for this erroneous classification is that the child saw the sheep in terms of that part of his cognitive structure (schema)[9] which seemed to apply. What this means is that the child at this point in his life having encountered four-legged woolly animals called dogs applied this schema to the sheep, since that was the only schema available to the child. McNally therefore argues that environmental events such as the above are assimilated into the cognitive structures not as a mechanistic transaction, but that the cognitive structures imposes its own organization, meaning or interpretation on an external stimulus. This means that the child compared the sheep to what was available in his cognitive structures for small woolly four-legged animals and since only dog was available interpreted it as dog. So assimilation is the intellectual process whereby the individual deals with the environment in terms of his present cognitive structures. This state of affairs however only exists until such time that accommodation in the cognitive structures of the child occurs. Accommodation occurs when the cognitive structures are forced to modify by the demands of the environmental event. In other words once the child have a more complete schema of a dog the child is forced to modify his cognitive structure in relation to the sheep. McNally states this as follows: "One very important feature of structures (schemas) is that they change as the result of the interaction of maturation and experience…"

This example makes it clear then that accommodation cannot proceed without assimilation and that assimilation and accommodation occur simultaneously. Anderson,

[9] Cognitive structures is intellectual structures that the individual has available for the interpretation and solution of problems posed by the environment.

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Reder and Simon **[1]** are of similar conviction as can be seen from the following quote: "A more careful understanding of Piaget would have shown that assimilation of knowledge also plays a critical role in setting the stage for accommodation…that the accommodation cannot proceed without assimilation." To emphasize this point let us take an extreme example. One cannot teach a six-year old child, who has just started school, the concept of a limit in mathematics since the child does not have a cognitive representation available to start an accommodation process from. In other words the child does not have the requisite mathematical knowledge to make a connection with the concept of limit. An important question is: Are learners of Euclidean geometry and proof in general in South Africa calling a sheep a dog because their cognitive structures have not been developed to maturation because of lack of ***experience*** and ***practice***?

Possible answers to this question are given in the previous section by the research done by the Russians and by the mental models theory which will be discussed later. The Russians discovered that in the first five grades learners mainly deal with Van Hiele level 1 activities, while from grade 6 learners suddenly had to deal with activities at a level 3 understanding. This implies that the learners did not go sequentially and hierarchically through the reasoning levels as required by Van Hiele. De Villiers **[29]** has pointed out that a similar situation exists in South Africa namely that grade 12 learners have only mastered Van Hiele level 2 or lower, whereas the examination requires mastery at level 3 and beyond. De Villiers indicated that most learners enter high school at Van Hiele level 2, but the axiomatic-deductive proofs belong to level 4. He also indicated that no provision is made to help learners to progress towards the required levels. As a consequence of the fact that learners did not go through the Van Hiele reasoning levels sequentially and hierarchically they therefore have not assimilated all the required knowledge necessary to build a complete cognitive structure required by the accommodation process. Stylianides and Stylianides is of similar conviction, this can be seen in the quote that we used earlier namely: "…several researchers have identified students' abrupt introduction to proof in high school as a possible explanation for the many difficulties that secondary school and university students face with proof, thereby proposing that students engage with proof in a coherent and systematic way throughout their schooling." As a result of the above arguments students find proof and proving at school level a very frustrating and difficult exercise.

The other part of the answer to the above question can be taken from theories concerning deductive reasoning. A relatively recent psychological theory of deductive reasoning is the mental models theory. Mental models theory refers to a representation in the mind that has a structure analogous to the structure of the situation it represents. This theory was developed by Johnson-Laird and hypothesizes about the process of deductive reasoning as it is applied to syllogisms[10]. This theory is described by Stylianides and Stylianides **[80]** as follows. The theory consists of three main stages. In the first stage, known as the comprehension stage, the reasoner constructs a mental model of the information presented in the premises of a syllogism. In the second stage known as the description stage the reasoner tries to devise a concise description of the model constructed in the first stage that concludes something not explicitly stated in the

[10] Syllogisms occur when one argues from premises to an inference or a conclusion.

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premises. The third stage known as the validation stage is where any essential deductive work is carried out. In this stage the reasoner also searches for counterexamples to the conclusion drawn in the second stage.

Stylianides et al **[80]** argue that the effectiveness of this model depends on a person's working memory[11] capacity. They argue further that limitation in working memory capacity results in errors in reasoning since people fail to consider all possible models of the premises that would provide them with counterexamples to the conclusions they have derived from their initial models. They use the example of Euclidean geometry to illustrate how working memory can be limited by too many separate sources of information. Sometimes when students are required to do proving in geometry they are provided with a diagram and a set of given information. To make sense of the two sources of information (diagrams and the givens) the students must mentally integrate the information. In other words read the givens, hold them in their working memory and then search the diagram for the appropriate places to apply it. Stylianides et al argue that this integration process is cognitively demanding and occupies a significant part of students' working memory capacity. To overcome errors in reasoning that occur as a result of too many pieces of separate information having to be processed at the same time, *practice* is suggested as a possible solution. Conversely Stylianides et al argue that practice can however also be associated with secondary aspects of student engagement with proof such as that of writing proofs in the two-column format. In such proofs the emphasis is on form and the result is that the proof becomes a ritual procedure and does not enhance the working memory capacity. The effective use of working memory and consequently deductive reasoning is therefore dependant on the reasoner freeing up the working memory by keeping essential information in the long term memory.

3.8 Physiology of learning

Recent discoveries of the workings of the human brain have opened up a host of new possibilities for teaching and learning. McGeehan **[65]**, for example, has the following opinion in this regard: "While scientists caution that they are only beginning to unravel the secrets of how humans learn, what they have already uncovered provides groundbreaking insights for educational practice. For the first time in the history of formal schooling, we have the opportunity and challenge to understand and act on the biology of learning rather than simply following traditional practices." What McGeehan means by biology of learning is that we should strive to understand what happens in the brain when learning takes place. She states this as follows: "Since students' mastery of the school curriculum happens primarily in the brain, it stands to reason that educators should be experts on the workings of that amazing organ." The question is what happens in the brain when learning takes place? The learning process in terms of what happens in the brain is explained by Sylwester **[82]** as follows: "From a cognitive perspective learning is explained as the building of neural connections[12]." The implications of this for pedagogy he explains as follows: "A familiarity of basic neural connectivity and brain

[11] Working memory is a system for temporarily storing and managing the information required to carry out complex cognitive tasks such as learning, reasoning, and comprehension.
[12] Neural connections are connections between neurons.

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structure leads to greater understanding of how the brain thinks, comprehends and ultimately learns." McGeehan **[65]** further elucidates this, as follows: "New experiences physically change the brain by causing neurons[13], the brain cells principally involved in cognition, to sprout new branches, or dendrites[14], and thus increase communication among neurons across microscopic gaps called synapses[15]. The synaptic leap of an electrical impulse between the axon[16] of one neuron and the dendrite of another is the physical basis of learning and memory. When a pathway of communication within a network of neurons is used repeatedly, it becomes increasingly efficient and we say that we have learned something."

[13] Neurons are a type of brain cell that receives stimulation from its branches, or dendrites, and communicates to other neurons by firing a nerve impulse along an axon.
[14]Dendrites are branch-like structures that extend from the neuron cell body and receive messages from other neurons.
[15] Synapses are the microscopic gap between the axon of one neuron and the dendrite of another.
[16] An axon is the part of a neuron that transfers a nerve impulse from the neuron cell body to a synapse with another cell.

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CHAPTER 4

THE INFLUENCE OF EMOTION, CONFIDENCE, EXPERIENCE AND PRACTICE ON THE LEARNING PROCESS

4.1 Introduction

A few years ago while I was teaching a grade 12 mathematics class I discovered that one of the girl learners, Shahieda[17] had major difficulties in learning mathematics. While teaching the class I would do some examples of a topic on the board and then request that the learners attempt one or more similar problems on their own. While the learners were attempting the problems I would walk around in the class and invariably I found that Shahieda was struggling. I would then sit next to her and point out the mistakes in her solution and then proceed to correct her argument. Subsequently I would request the class to attempt another problem of the same kind. Shahieda would then provide the correct solution to this follow-up problem. Satisfied that she was now cognizant of the requirements of such problems I would proceed to the next topic.

To my amazement when a test was written that covered the same topic Shahieda would make exactly the same mistakes that I had pointed out to her. Her results in the two exams of that year were even worse. She scored less than 10% for both the first and second paper of both exams. This was despite my numerous attempts to get her to reason correctly. What puzzled me was that in the class she would seem to understand and even provide correct arguments, but as soon as she wrote tests or exams she would revert back to her erroneous arguments. I therefore came to the conclusion that factors other than the normal reasoning problems were responsible for this behaviour.

4.2 The influence of emotion

An intriguing question is how does the human brain deal with new incoming information? How the human brain deals with incoming information is explained by McGeehan **[65]** as follows: "First the sensory stimuli hit the neurons in the appropriate sensory cortex[18]. These crude sensations are then relayed through the thalamus[19] and sent to the sensory association area of the neocortex where they are put together into objects we recognize. Next (and almost simultaneously) the information is sent to the amygdala[20] for emotional evaluation and to the frontal cortex for content evaluation. On the basis of its analysis of physical features of the stimuli, the brain begins to construct meaning."

Emotions can affect learning, in both a positive and negative way. Kort et al **[55]** confirm this in the following quote: "When a learner experiences positive emotions, the learning

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[17] Not her real name

[18] The cortex is a neuron-packed outer layer of the brain in which conscious thought takes place.

[19] The thalamus is a sensory relay station located deep within the middle of the brain.

[20] The amygdala is an almond –shaped structure in the middle of the brain, connected to the hippocampus, which detects the emotional content of sensory data and plays a role in the formation of emotion-laden memories.

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process can be enhanced. When a learner experiences negative emotions, the learning process can be disabled." Goleman **[39]** is of similar opinion as the following quote shows: "Students who are anxious, angry, or depressed don't learn; people who are caught in these states do not take in information efficiently or deal with it well." The importance of the effect of emotion on the learning process is shown in the following quote from Sylwester **[82]**: "Emotion drives attention, which drives learning, memory and problem solving and almost everything else we do …by not exploring the role that emotion plays in learning and memory, our profession has fallen decades behind in devising useful instructional procedures that incorporate and enhance emotion." Sylwester further emphasizes this as follows: "Far more neural fibers project from our brain's emotional center into the logical/ rational centers than the reverse, so emotion is often a more powerful determinant of our behaviour than our brain's logical/ rational processes."

Incoming information also needs to have personal meaning and emotional importance for students as a prerequisite to being stored in the long term memory. In other words students tend not to remember for long those things that lack personal meaning and emotional importance for them. This is confirmed in the following quote of McGeehan **[65]**: "…when information lacks personal meaning and an emotional hook, the neural networks needed to create long-term memories are not formed." It is clear therefore that emotion plays a crucial role in the learning process.

When I was at primary school some of the mathematics teachers used to call learners to the board to solve mathematics problems on the board. The teacher would stand behind the learner with a cane and as soon as the learner made a mistake the teacher would beat the learner. Another popular method was where the teacher would do mathematics revision with a cane. Any learner that could not supply an answer to a question would routinely receive a beating. I think the teacher's motivation for using these methods was that the teacher was under the impression that the learners were lazy and that the beatings and resulting pain would inspire them to become industrious and would cause them to learn and understand. The strongest emotions that these learners therefore came to associate with mathematics is one of paralyzing fear, humiliation and also that not understanding is a bad thing. The sad part is that neuroscience has shown that memories that are strongly charged with emotion are most likely to go into the long term memory. So these learners that had a traumatic experience with mathematics will tend to remember it for a long time. The result of this is that since all the previous experiences of these learners with mathematics have been bad they expect that their next experience with the learning of mathematics would also be bad. This then becomes a self-fulfilling prophecy. In other words the learner expects a negative experience because the emotions in their long term memory prepare them for a bad experience and hence they get a bad experience. Such learners will struggle with mathematics not necessarily because they do not understand it, but because their previous experience dictates that it must be so. Hence contrary to their aim of teaching the learners to learn and understand, these teachers instead taught them anxiety, fear, humiliation and confusion.

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Even teachers that do not use intimidation as a method of teaching and that have good intentions sometimes do not take into consideration the effect of emotion on the learning process. This is starkly illustrated in the following quote from Kort et al **[55]**: "When teachers present material to the class, it is usually in a polished form that omits the natural steps of making mistakes (e.g. feeling confused) recovering from them (e.g. overcoming frustration), deconstructing what went wrong (e.g. not becoming dispirited), and starting over again (with hope and enthusiasm). Those who work in science, math, engineering, and technology (SMET) as professions know that learning naturally involves failure and a host of associated affective responses." Kort et al therefore conclude that what we fail to teach students is that these feelings associated with various levels of failure are normal parts of learning, and that they can actually be helpful signals for how to learn better. It is evident therefore that instructors in mathematics should be cognizant of the emotional requirements of their subject and should include examples that allow students to experience a variety of emotions in the learning process so as to prepare the student for the emotional rigours involved in doing mathematics.

4.3 The influence of Confidence

It is my contention that in order to become proficient in theorem proving (or any mathematical process) students need to be confident in what they are doing. The literature is commensurate with my view of the connection between confidence and success in mathematics. Burton **[9]** for example states: "This teacher is reflecting a widely held view that performance in mathematics and confidence go hand-in-hand. Success in mathematics breeds confidence. Confidence in mathematics breeds success." Research done in Canada by PISA[21] **[10]** found that a student's self-confidence and level of anxiety about mathematics were strongly associated with their performance. They found that those with high levels of confidence in their ability to learn mathematics performed much higher than those with low levels. Conversely students with a high level of anxiety about mathematics, such as feelings of helplessness or stress when dealing with mathematics, performed much lower than students with less anxiety. Clute **[15]** is of the same opinion as can be seen from the following quote: "Hence, people high in mathematics confidence perform better on mathematical tasks. Mathematics anxiety is strongly but negatively related to mathematics confidence."

In order to inform our discussion on the relationship between confidence and success in mathematics, I think it is imperative that we define what we mean when we are referring to confidence. Although there are different views as to the meaning and definition of confidence in mathematics, our understanding of mathematical confidence will be informed by that of Burton **[9]** which is as follows: "…I saw confidence as a label for a confluence of feelings relating to beliefs about the self and about one's efficacy to act within a social setting, in this case the mathematics classroom." Since the two main protagonists in the mathematics classroom are the teacher and the student we therefore

[21] PISA (Programme for International Student Assessment) is a collaborative effort among member countries of the Organisation for Economic Co-operation and Development (OECD) – this program is designed to regularly assess the achievement of 15-year-olds in reading, mathematical and scientific literacy using a common international test.

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have to ascertain their views on confidence. The study done by Burton found that teachers regarded confidence as individual and behavioural. That is the teachers did not think that confidence in mathematics involved a social act and that confidence in the individual is exhibited by behaviour such as willingness to answer or to attempt. Conversely the students associated confidence with feelings and how the classroom could function to make those feelings better or worse. The students were in favour of a collaborative working style and were of the opinion that getting answers correct fueled the confidence level. Furthermore they were convinced that both knowledge and understanding contributed to confidence.

If confidence is as crucial as the literature suggests an important question then would be what do teachers have to do in order to inculcate their students with confidence in mathematics? I am of the opinion that some of the student responses of the Burton **[9]** study provide an answer to this question. These include the following: "They wanted teachers to facilitate discussion, teamwork, a light-hearted approach, a relaxed classroom where you are not afraid of making errors…they did not want to be put down, persistently asked the same question, made to look a fool or feel patronized, put into a position where others laugh at you, …the students felt that teachers should explain well, should not rush the work, should know what they are talking about and should be sensitive to students who are struggling to understand." Clute **[15]** on the other hand was of the opinion that instructional strategy plays a crucial role in mathematics achievement. The study done by Clute indicates that students with low levels of anxiety and hence high confidence were better served by the discovery method of teaching[22]. Conversely students with a high level of mathematical anxiety and low confidence relied heavily on a well-structured, controlled plan for learning.

4.4 The influence of Experience

Another question that deserves our attention is: Is there a relationship between confidence and experience? Anderson, Reder and Simon **[1]** provide the following answer to this question: "Cognitive competence (in this case mathematical competence) depends on the availability of symbolic structures (e.g. mental patterns or mental images) that are created in response to experience." The answer provided by neuroscience is much more compelling. Findings from brain research indicate that intelligence is a function of experience. McGeehan **[65]** explains it as follows: "New experiences physically change the brain by causing neurons, the brain cells principally involved in cognition, to sprout new branches, or dendrites, and thus increase communication among neurons across microscopic gaps called synapses…The findings of neuroscientists affirm the importance of experience in the development of dendrites and, by extension, in the results of this development which we call learning and observe as intelligence." Clute **[15]** has the following opinion about the relationship between confidence and the learning of mathematics: "If one lacks confidence in one's ability to perform mathematical tasks, it seems reasonable to conclude that there is a lack of respect for or trust in one's own

[22] The discovery method of teaching was devised by Clute for her study. In this method the focus is on the teacher interacting with students to develop subject matter concepts from which the students then "discover" the answer.

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instincts or judgments when it comes to learning mathematics." Therefore since confidence influences the learning of mathematics and in turn learning is influenced by experience it is clear that experience does influence confidence. I would therefore suggest that the more experienced one becomes in mathematical procedures and techniques the more your confidence will be enhanced. An important next question then, is how does one gain experience in mathematics?

4.5 The influence of Practice

Bransford et al **[7]** uses chess to provide an answer to the above question. They argue that it is estimated that world-class chess masters require from 50 000 to 100 000 hours of practice to reach that level of expertise. Much of the practice time involves the development of pattern recognition skills that support the fluent identification of meaningful patterns of information plus knowledge of their implications for future outcomes. Bransford therefore quote Singley and Anderson to make the point that in all domains of learning, the development of expertise occurs only with major investments of time, and the amount of time it takes to learn material is roughly proportional to the amount of material being learned. There is a belief among many people that talent determines who becomes an expert in a particular area of learning. Ericsson et al **[35]** however have found that even seemingly talented individuals require a great deal of practice in order to develop their expertise. It is clear therefore that in order to gain experience one has to do extensive practice. Anderson, Reder and Simon **[1]** states for example that the last 20 years of research on cognitive psychology showed that real competence only comes with extensive practice. They state: "The instructional task is not to "kill" motivation by demanding drill, but to find tasks that provide practice while at the same time sustaining interest." Vygotsky **[89]** was of similar opinion and writes in this regard: "Wundt long ago established that the latent period of a complex reaction decreases with practice."

4.6 Negotiating the learning process

Based on the above arguments the following is our view as to some of the things that contribute towards making a student successful in the study of mathematics. First the teacher has to endeavor to create an atmosphere in the classroom that students find non-threatening. In order to become confident, the student needs to practice what he /she has been taught. This is the way it works: the student in the class has his/her zone of proximal development (zpd)[23] extended by his/her teacher or more capable peer. In other words the student is helped by the teacher or peer to do problems that the student could not master on his/her own. This is because according to Vygotsky **[89]** what is in the zone of proximal development today will be the actual developmental level tomorrow. In order to assimilate this (i.e. to make it part of his/her knowledge structures) the student now needs to practice. This has to be done by the student on his/her own. This also requires the student to invest a lot of time into practice. MGeehan **[65]** confirms this as follows:

[23] The zone of proximal development is the distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers.

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"When a pathway of communication within a network of neurons is used repeatedly, it becomes increasingly efficient and we say that we have learned something."

What this also requires is a commitment on the side of the student which requires the student to have a thirst for this kind of knowledge i.e. there must be another reason besides studying for tests or exams to master the required concepts in mathematics. That is the student must have an emotional connection with the knowledge i.e. a hunger for knowledge for knowledge's sake. This then becomes a cycle i.e. a student is extended beyond his/her zone of proximal development then he/she consolidates this by practicing and hence become more confident as he/she masters the subject matter. This confidence therefore is fuelled by being successful in attempting the subject matter and hence building more experience.

Assessment will help the teacher identify which reasoning problems students still have. So after assessing students the teacher should try and identify, based on their mistakes, where the students still have gaps or faults in their reasoning. The teacher should then devise strategies to help the student in overcoming these problems with their reasoning. Once the teacher is satisfied that the student is now reasoning correctly, the student should be reassessed to confirm that there are no more gaps or faults in their reasoning. The fact that the student will now be able to successfully solve these kinds of problems will further boost his/her confidence and at the same time provide the student with valuable experience.

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CHAPTER 5

RESEARCH METHODOLOGY

5.1 Introduction

The students of the MAM 112 course (a first year elective mathematics course) at UWC were utilized by us as the experimental group. At the start of the semester 39 students were enrolled for this course. For various reasons seven of these students dropped out. Six of the remaining 32 were students that repeated the course and therefore were not taken into consideration for the statistical analysis of the study. The students of the MAM 111 course (a first year core module at UWC) formed the control group of the study. Only 21 of these students completed both the pre and post-test of the logic unit and therefore only the results of these students were taken into consideration for statistical analysis.

Our intention was to complete the study during the first semester of 2007. However we could not finish all the content in the first semester and therefore had to use 3 to 4 weeks of the second semester as well. The total amount of learning hours available was 150 hours, 96 hours of which was utilized as contact time with students. The contact sessions consisted of four one hour lectures and a two hour tutorial session per week. The experimental group received teaching on logic and thereafter teaching on the different types of proof whereas the control group received no teaching on logic but were taught on topics of Differential Calculus. The textbook that was used by the experimental group was "Discrete Mathematics with Applications" by Susanna S. Epp **[33]**.

The study used a pre-test post-test design. The measuring instruments for this study were mainly these pre- and post-tests. Only for the first logic puzzle an experimental-control group design was used. For all the logic puzzles the pre-test was given without the students having received any teaching on the given topic. The students received no feedback on the pre-test. After the pre-test was administered, the experimental group was taught the logic skills involved in the puzzles. A post-test was then administered to determine if students had acquired the requisite skill. The experimental group of students also wrote class tests that counted towards their course mark but these tests were not utilized as a measuring instrument in the study. Conversely the pre- and post-tests did not count towards the course mark, but formed part of the measuring instruments of the study. The tests were administered during the regular class meetings. Although both quantitative and a qualitative designs were utilized in the study, the majority of the findings were based on a quantitative design.

During their very first lecture both the control and experimental groups were asked to solve a logic puzzle of the Lewis-Carol type. At this stage none of the students had received any formal instruction in mathematical logic. The reason why we gave the students pre-tests on the logic puzzles without first teaching them is based on an experiment of Piaget. In this experiment Piaget asks a five-year –old child why the sun does not fall. What he is assuming, according to Vygotsky **[89]**, is that the child does not

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have a ready answer for such a question nor the capabilities to generate one. Furthermore the point of asking a question that is so far beyond the reach of the child's intellectual skills is to eliminate the influence of previous experience and knowledge. So the aim is to obtain the tendencies of children's thinking in "pure form" i.e. thinking that is entirely independent of learning. This pure form of thinking is what we wanted to determine of our students i.e. what their current cognitive abilities were in terms of deductive reasoning.

5.2 Learning theory

We support the information-processing learning theory of constructivism. This information –processing approach is based on an approach espoused by Anderson, Reder and Simon **[1]**. According to this approach learning is viewed as an active, constructive process in which students attempt to resolve problems that arise as they participate in the mathematical practices of the classroom. Our teaching therefore was designed to embrace this kind approach.

5.3 Teaching Methodology

The teaching methodology was discussion-based with problems discussed and solved during lecture periods. Typically the lecturer would do some examples on the board, which would then be discussed. The discussion normally was preceded by the lecturer asking pertinent questions to determine if the students understood the examples. If it became clear that students did not understand the examples the lecturer would then initiate a discussion based on student questions. Once we were satisfied that the majority of students were aware of the cognitive requirements of the examples we posed problems similar to the examples to students. These problems would then be attempted by students and if necessary with help from more capable peers, the teaching assistant or the lecturer. Sometimes the lecturer would ask one of the students to do his/her solution on the board. The student was expected to not only write down the solution on the board, but also to explain his/her solution. These explanations gave us insight into the students' understanding and reasoning. These solutions were also discussed and gaps or errors in reasoning were pointed out and corrected immediately. This was done in order to consolidate what the students have learned. This consolidation exercises were given immediately after the lesson and was mostly done in class. The importance of this approach is emphasized by Kutzler **[58]** in the following quote: "In the psychology of learning, scientists discovered the concept of reinforcement and showed that reinforcement works best if it follows the action immediately". At the end of the lecture students would be given exercises to complete as homework. The solutions to the homework were provided in the subsequent lecture.

5.4 The logic component of the course

As already indicated the first logic puzzle was given to the students in the very first lecture, before they had any exposure to teaching on the subject of logic and therefore this puzzle was used to test the initial deductive abilities of students. This puzzle was

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named puzzle I and was given as the first pre-test. Puzzle I contained compound and conditional statements. The puzzle is presented in Appendix A1.

The following day the students were given another puzzle, i.e. puzzle 2. This puzzle known as the knights and knaves puzzle contained only two statements and a question. The solution to the puzzle was based on contradiction. The students were given this puzzle without prior teaching on contradiction and therefore this puzzle formed pre-test 2. This pre-test was used to ascertain the students' level of comprehension of the use of contradiction as a method of proof. No immediate feedback was provided to students as to the solution of the puzzle. The puzzle is presented in Appendix B1. Students were required to write their names on the puzzles so that we could track each student individually.

Both the experimental and control group students received no feedback on puzzle I. In the subsequent lectures the experimental group received teaching on logic. We started with the logic of compound statements. In propositional logic the words sentence, true and false are the initial undefined terms. We then proceeded to definitions of statements and compound statements and the symbolism involved in these and thereafter to truth tables. This was followed by an investigation of the logical equivalence of statements which in turn was followed by conditional statements. Conditional statements were followed by discussions on valid and invalid arguments. The logic component of the course was ended with teaching on the logic of quantified statements.

5.4.1 Compound Statements

Statements are defined as a sentence that is either true or false but not both. A compound statement is formed when statements are joined by the logical connectives: *and* $(\wedge)$, *or* $(\vee)$ and *not* $(\sim)$.

5.4.1.1 Truth tables

The truth table for a given statement form displays the truth values that correspond to the different combinations of truth values for the statement variables. For a statement to have well-defined truth-values means that it must either be true or false, but not both. So if p is a statement then the negation of p is "not p" and in symbol form is given as "$\sim p$". This means that if p is true then $\sim p$ is false and conversely if p is false then $\sim p$ is true. The truth table is shown in table 1 below (T = true and F = false).

Truth table for $\sim p$

p	$\sim p$
T	F
F	T

Table 1

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The conjunction of statements p and q is "*p* and *q*" denoted $p \wedge q$. It is true when and only when both p and q are true. If either p or q is false, or if both are false, $p \wedge q$ is false. The truth table is shown in table 2 below.

Truth table for $p \wedge q$.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 2

The disjunction of statement variables *p* and *q* is "*p* or *q*" denoted $p \vee q$. It is true when at least one of *p* or *q* is true and is false only when both *p* and *q* are false. The truth table is shown below in table 3.

Truth table for $p \vee q$.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 3

To evaluate the truth values of more general statements the following steps must be adhered to. First evaluate the expressions within the innermost parentheses, then evaluate the expressions within the next innermost set of parentheses, and so forth until you have the truth values for the complete expression. An example of such a statement is: $(p \vee q) \wedge \sim(p \wedge q)$. The truth table for this statement is given in table 4 below.

Truth table for $(p \vee q) \wedge \sim(p \wedge q)$

p	q	$p \vee q$	$p \wedge q$	$\sim(p \wedge q)$	$(p \vee q) \wedge \sim(p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Table 4

The logical equivalence of statements was investigated next.

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5.4.1.2 Logical equivalence of statements

Two statement forms are called logically equivalent if, and only if, they have identical truth values for each possible substitution of statements for their statement variables. The logical equivalence of statement forms A and B is denoted by $A \equiv B$.
A tautology is a statement form that is always true regardless of the truth values of the individual statements substituted for its statement variables.
A contradiction is a statement form that is always false regardless of the truth values of the individual statements substituted for its statement variables. The statement form $p \vee \sim p$ is a tautology and $p \wedge \sim p$ is a contradiction.

The following is a summary of the logical equivalences that can be used to simplify statement forms: Given any statement variables *p, q* and *r* a contradiction *c* and a tautology *t*, the following logical equivalences hold:

1.	Commutative laws:	$p \wedge q \equiv q \wedge p$ and $p \vee q \equiv q \vee p$
2.	Associative laws:	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ and $(p \vee q) \vee r \equiv p \vee (q \vee r)$
3.	Distributive laws:	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ and $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4.	Identity laws:	$p \wedge t \equiv p$ and $p \vee c \equiv p$
5.	Negation laws:	$p \vee \sim p \equiv t$ and $p \wedge \sim p \equiv c$
6.	Double negative laws:	$\sim (\sim p) \equiv p$
7.	Idempotent laws:	$p \wedge p \equiv p$ and $p \vee p \equiv p$
8.	De Morgan's laws:	$\sim (p \wedge q) \equiv \sim p \vee \sim q$ and $\sim (p \vee q) \equiv \sim p \wedge \sim q$
9.	Universal bound laws:	$p \vee t \equiv t$ and $p \wedge c \equiv c$
10.	Absorption laws:	$p \vee (p \wedge q) \equiv p$ and $p \wedge (p \vee q) \equiv p$
11.	Negation of *t* and *c*:	$\sim t \equiv c$ and $\sim c \equiv t$

The following is an example where the above laws are used to verify a logical equivalence: $\sim ((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q) \equiv p$

Solution

$\sim ((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q)$
$\equiv \sim (\sim p \wedge (q \vee \sim q)) \vee (p \wedge q)$ - by the distributive law
$\equiv \sim (\sim p \wedge t) \vee (p \wedge q)$ - by the negation law
$\equiv \sim (\sim p) \vee (p \wedge q)$ - by the identity law
$\equiv p \vee (p \wedge q)$ - by the double negative law
$\equiv p$ - by the absorption law

Upon the completion of this section a formal test was given which covered the topics of compound statements, negation of a statement, conjunction and disjunction, truth tables and logical equivalences. This test counted towards the course mark, but was not used as a measuring instrument.

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5.4.2 Conditional Statements

A conditional statement is of the form if *p* then *q,* where p and q are statements (*p* is called the hypothesis and *q* is called the conclusion). This is based on the fact that when you make a logical inference or deduction you reason from a hypothesis to a conclusion. Sentences such as these are called conditional because the truth of statement *q* is dependent on the truth of statement *p*. If *p* then *q* is denoted by $p \rightarrow q$. It is false when p is true and q is false, otherwise it is true. The truth table for the conditional statement is shown in table 5.

Truth table for $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 5

In expressions that include $\rightarrow$as well as other logical operators such as $\wedge, \vee$ and $\sim$ the order of operations is that $\rightarrow$ is performed last.

The negation of "if *p* then *q*" is logically equivalent to "*p* and not *q*". Students were provided with exercises where they were required to symbolically show equivalence as well as instances where they were required to rewrite the negations of if-then statements in everyday English. This was done throughout the logic component and was done to prevent students getting too mechanistic in their reasoning when applying rules of logic.

5.4.2.1 Contra-positive, Converse and Inverse of a conditional statement

A conditional statement is logically equivalent to its contra-positive. The contra-positive of a conditional statement of the form "if *p* then *q*" is "if ~*q* then ~*p*". The fact that a conditional statement is equivalent to its contra-positive is one of the most fundamental laws of logic.
The converse of "if *p* then *q*" is "if *q* then *p*". The inverse is "if ~*p* then ~*q*". Both the converse and inverse of a conditional statement is not logically equivalent to the statement.

5.4.2.2 Bi-conditional

Given statement variables *p* and *q*, the bi-conditional of *p* and *q* is "*p* if, and only if, *q*" and is denoted $p \leftrightarrow q$. It is true if both p and q have the same truth values and is false if *p* and *q* have opposite truth values. The truth values for the bi-conditional is shown in table 6 below.

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Truth table for $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 6

The hierarchy of operations for the five logical connectives are:

1. $\sim$
2. $\wedge, \vee$
3. $\rightarrow, \leftrightarrow$

5.4.2.3 Necessary and sufficient conditions

If r and s are statements, then *r is a sufficient condition for s* means "if r then s". On the other hand *r is a necessary condition for s* means "if not r then not s"

Upon completion of this unit the students were tested on negations, converse, inverse, contra-positive, necessary and sufficient conditions and rewriting in if-then form. This test contributed to their course mark, but was not used as part of the measuring instrument.

5.4.3 Valid and invalid argument forms

An argument is a sequence of statements. All statements except the final are called premises or hypotheses. The final statement is known as the conclusion. The symbol $\therefore$, read "therefore" is placed just before the conclusion.

What does it mean to say that an argument form is valid? To say that an argument form is valid means that no matter what particular statements are substituted for the statement variables in its premises, if the resulting premises are all true, then the conclusion is also true. When an argument is valid and its premises are true, the truth of the conclusion is said to be inferred or deduced from the truth of the premises. Truth tables can be utilized to test the validity of an argument. In order to do this one has to construct a truth table that shows the truth values of all the premises and the conclusion. In each critical row (rows in which all the premises are true) determine whether the conclusion of the argument is also true. If in each critical row the conclusion is also true, then the argument form is valid. If there is at least one critical row in which the conclusion is false the argument form is invalid. The following is an example where a truth table is used to determine if an argument is valid: $p \vee (q \vee r)$

$$\begin{array}{l} p \vee (q \vee r) \\ \sim \mathrm{r} \\ \therefore p \vee q \end{array}$$

The truth table is shown in table 7

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Truth table

p	q	r	$q \vee r$	$p \vee (q \vee r)$	$\sim$r	$p \vee q$
T	T	T	T	T	F	T
T	T	F	T	T	T	T
T	F	T	T	T	F	T
T	F	F	F	T	T	T
F	T	T	T	T	F	T
F	T	F	T	T	T	T
F	F	T	T	T	F	F
F	F	F	F	F	T	F

Table 7

The rows that are highlighted are the critical rows. Since in each row where the premises are both true (critical rows) the conclusion is also true, we conclude that the argument is valid.

5.4.3.1 Rules of inference

There are rules in logic which state that certain arguments are valid. These rules are known as the rules of inference. The rules are given below:

1. One of the best known rules of inference is ***modus ponens***. Modus ponens in Latin means method of affirming. Its argument form is as follows:

   If p then q
   p
   $\therefore q$

2. **Modus Tollens** means method of denying. Its argument form is:

   If p then q
   $\sim q$
   $\therefore \sim p$

3. **Disjunctive addition** is used for generalization and has the following valid argument form:

   p or q
   $\therefore p \vee q$ $\therefore p \vee q$

4. **Conjunctive simplification** is used for particularization and has the following argument form:

   $p \wedge q$ or $p \wedge q$
   $\therefore p$ $\therefore q$

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5. **Disjunctive syllogism** is the valid argument form that says when you have only two possibilities and you can rule out one, the other must be the case. Its argument form is:

$$\begin{array}{lcl} p \vee q & \text{or} & p \vee q \\ \sim q & & \sim p \\ \therefore p & & \therefore q \end{array}$$

6. Many arguments in mathematics contain chains of if-then statements. If one statement implies a second and the second implies the third, then you can conclude that the first statement implies the third. This valid argument form is known as ***hypothetical syllogism***. Its symbolic argument form is:

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \therefore p \rightarrow r \end{array}$$

7. It often happens that one knows that one thing or another is true. If you can show that in either case a certain conclusion follows, then this conclusion must also be true. This is known as ***Dilemma: proof by division into cases***. Its valid argument form is:

$$\begin{array}{l} p \vee q \\ p \rightarrow r \\ q \rightarrow r \\ \therefore r \end{array}$$

8. **Rule of contradiction** – suppose p is some statement whose truth you wish to deduce. If you can show that the supposition that statement p is false leads logically to a contradiction, then you can conclude that p is true. The contradiction rule forms the logical base of the method of proof by contradiction. Its argument form is as follows:

$$\begin{array}{l} \sim p \rightarrow c \ [where\ c\ is\ a\ contradiction] \\ \therefore p \end{array}$$

5.4.3.2 Fallacies

A fallacy is an error in reasoning that results in an invalid argument. Two types of fallacies that were discussed were the converse and inverse error. These are arguments that resemble modus ponens and modus tollens but are invalid.

The general form of the converse error is as follows:

$$\begin{array}{l} p \rightarrow q \\ q \\ \therefore p \end{array}$$

This argument form is invalid since a conditional statement is not logically equivalent to its converse.

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The form of the inverse error is as follows:

$$p \to q$$
$$\sim p$$
$$\therefore \sim q$$

This argument form is invalid because a conditional statement is not logically equivalent to its inverse.

Students were also made aware of valid arguments that have false conclusions and invalid arguments that have true conclusions.

5.4.3.3 Examples based on argument forms

Upon the completion of the section on valid and invalid arguments students were provided with a list of the rules of inference. They were then required to deduce conclusions from given premises providing reasons for each deduction. The following is an example of such an exercise:

Example 1

Use the rules of inference to deduce the conclusion from the premises, giving a reason for each step:

1. $\sim p \vee q \to r$
2. $s \vee \sim q$
3. $\sim t$
4. $p \to t$
5. $\sim p \wedge r \to \sim s$

$\therefore \sim q$

Solution

6.	$\sim p$	from 3 & 4 – modus tollens
7.	$\sim p \vee q$	from 6 – disjunctive addition
8.	r	from 1 & 7 – modus ponens
9.	$\sim p \wedge r$	from 6 & 8 – conjunctive addition
10.	$\sim s$	from 5 & 9 – modus ponens
11.	$\sim q$	from 10 & 2 – disjunctive syllogism

Students were also provided with exercises where they were required to rewrite statements in symbol form and then use the rules of inference to deduce a conclusion. The following is an example of such an exercise:

Example 2

You are about to leave for school in the morning and discover you don't have your glasses. You know the following statements are true:

(a) *If my glasses are on the kitchen table, then I saw them at breakfast.*

(b) *I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.*

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(c) *If I was reading the newspaper in the living room, then my glasses are on the coffee table.*
(d) *I did not see my glasses at breakfast.*
(e) *If I was reading my book in bed, then my glasses are on the bed table.*
(f) *If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.*
Where are the glasses?

Solution

Let p = my glasses are on the kitchen table
q = I saw my glasses at breakfast
r = I was reading the newspaper in the living room
s = I was reading the newspaper in the kitchen
t = my glasses are on the coffee table
u = I was reading my book in bed
v = my glasses are on the bed table

The statements (a) to (f) then translate as follows:
(a) $p \to q$
(b) $r \vee s$
(c) $r \to t$
(d) $\sim q$
(e) $u \to v$
(f) $s \to p$

Based on this the following deductions can now be made:
1. $\sim p$ from (a) & (d) – modus tollens
2. $\sim s$ from (f) & 1 – modus tollens
3. r from (b) & 2 – disjunctive syllogism
4. t from (c) & 3 – modus ponens

Hence t is true and the glasses are on the coffee table.

5.4.3.4 Applying the Contradiction Rule

Students were also provided with exercises where they were required to apply the rule of contradiction. The students were given such a puzzle as a pre-test i.e. where they were required to solve a knights and knave puzzle without prior teaching on the rule of contradiction. This pre-test therefore was used to ascertain the students' level of comprehension of the use of contradiction as a method of proof. A tutorial session was utilized to discuss the solution to this puzzle. After discussing the solution to the puzzle students were presented with an equivalent puzzle that was used as a discussion exercise. This was done to illustrate the cognitive processes involved in getting to a solution with this type of problem. Students were then given a post-test[24] to determine if they had progressed in their deductive abilities where method of contradiction is concerned. The

[24] See Appendix B2.

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results of the post-test however showed that approximately 50% of the students still had problems with the concept of contradiction. Consequently we decided that an intervention session was necessary. Another tutorial session was utilized for this purpose.

I am of the opinion that teaching strategy plays a crucial role in student understanding of the application of certain tools in mathematics. An example of this is the teaching strategy one can employ to connect the rule of contradiction to the method of proof by contradiction. Our aim was to use the knights and knaves puzzles as a precursor to proof by contradiction.

Epp **[33]** states the following about proof: "Probably the most important reason for requiring proof in mathematics is that writing proof forces us to become aware of weaknesses in our arguments and in the unconscious assumptions we have made." The teaching strategy employed by the lecturer to eliminate reasoning errors of students in proof by contradiction, is illustrated in the following succinctly described teaching session. The students were supplied with the following knights and knaves puzzle which they had to solve:

The logician Raymond Smullyan describes an island containing two types of people: knights who always tell the truth and knaves who always lie. You visit the island and are approached by two natives who speak to you as follows:

A says: Both of us are knights
B says: A is a knave.

What are A and B?

The lecturer called Marsha to present her answer on the board.

5.4.3.5 Marsha's solution:

Suppose A is a knight, then what he says is true, then they both are knights, but B says that A is a knave $\therefore$ *it's a contradiction because A cannot be both a knight and a knave* $\therefore$ *A is a knave and B is a knight.*

The lecturer then asked the class if they were satisfied with this answer. (Some of the students' answers he ignored since these answers were not helping to improve their understanding of contradiction). Thulani stated that he does not see how Marsha got to the conclusion that B is a knight. Siphokazi stated that the conclusion and argument do not connect. Using these two answers the lecturer then proceeded to make certain that the whole class understood that Marsha did not show how she got from "A and B cannot be both a knight and a knave" to "$\therefore$ A is a knave and B is a knight". The lecturer also explained that Marsha's answer was one long sentence and that one is required to make a supposition and then make a full stop. One then has to think about what the supposition implies. The answer to this forms the next statement in the argument. The next statement is then analyzed for its implications, etc. The aim therefore was to get the students to reason one step at a time. Using the above argument and input from the students the

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lecturer then rectified Marsha's answer on the board so that it reflected the correct answer.

5.4.3.6 Marsha's corrected answer

1. Suppose A is a knight.
2. Then what A says is true – by definition, of knights
3. Therefore B is a knight
4. Therefore what B says is true – by definition of knights
5. Then A is a knave
6. So A is both a knave and a knight
7. This is a contradiction
8. Hence our supposition is false
9. $\therefore$ A is a knave – negation of supposition
10. $\therefore$ What B says is true
11. $\therefore$ B is a knight – by definition of knight.

The lecturer then asked the class to explain the structure of proof by contradiction. The following is Earl's explanation:
"You first make a supposition, which will have consequences. These consequences will lead to a contradiction which proves the supposition false. Which makes the opposite of the supposition true."

After this and other encouraging responses from the students we were satisfied that the majority of the students now had a reasonable good grasp of the reasoning involved in proof by contradiction and that we could now proceed to the next topic of discussion.

5.4.4 Quantified statements

Analysis of compound statements can elucidate many aspects of reasoning, but there are many cases in everyday life and mathematics where it cannot be used to determine validity. Hence a different methodology is needed to deal with such cases. The logic of quantified statements provides a means to deal with arguments that cannot be analyzed as before.

Arguments where sentences need to be separated into parts and where words such as "all" or "some" play a special role are classified as quantified statements in logic. In normal English grammar declarative sentences can be separated into subjects and predicates. Here predicate refers to the part of the sentence that gives information about the subject. In logic, however a predicate is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. The domain of a predicate variable is the set of all values that may be substituted in place of the variable. The set of all elements that make the predicate true is called the truth set of the predicate. The truth set of a predicate *P(x)* is denoted by: $\{x \in D / P(x)\}$

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Quantifiers are words that refer to quantities such as "some" or "all" and tell for how many elements a given predicate is true. The symbol $\forall$ denotes "for all" and is called the universal quantifier.

Let Q(x) be a predicate and D the domain of x. A universal statement is a statement of the form " $\forall\, x \in D,\ Q(x)$ " It is defined to be true if, and only if, Q(x) is true for every x in D. It is defined to be false if, and only if, Q(x) is false for at least one x in D. After the above theory on universal quantifiers was discussed students were given exercises that contained mathematical examples of applications of the universal quantifier.

The existential quantifier is denoted by the symbol $\exists$ and means "there exists". Let Q(x) be a predicate and D the domain of x. An existential statement is a statement of the form " $\exists\, x \in D \text{ such that } Q(x)$ " It is defined to be true if, and only if, Q(x) is true for at least one x in D. It is false if, and only if, Q(x) is false for all x in D. After the existential quantifiers were discussed students were presented with exercises that dealt with mathematical examples of existential statements.

Since it is important to be able to translate from formal into informal language when trying to make sense of mathematical concepts we provided the students with exercises that required them to translate universal statements from formal into informal language and vice versa.

Since a great many of mathematical statements are universal conditional statements of the form: $\forall\, x, \text{ if } P(x) \text{ then } Q(x)$, it is imperative that students are familiar with statements of this form. We therefore required students to translate universal conditional statements from formal to informal language and vice versa.

Equivalent forms of universal and existential statements were then tackled. The negation of a universal statement "all are" is logically equivalent to an existential statement "some are not". In formal language it can be given as follows: $\sim\left(\forall\, x \in D,\ Q(x)\right) \equiv \exists\, x \in D \text{ such that } \sim Q(x)$
Conversely the negation of an existential statement "some are" is logically equivalent to a universal statement "all are not". Symbolically it can be given as follows:
$\sim\left(\exists\, x \in D, \text{ such that } Q(x)\right) \equiv \forall\, x \in D,\ \sim Q(x)$
Examples of both forms of negation using both formal and informal cases were discussed and analyzed in class.

To determine the understanding of the students regarding quantified statements a test was administered. This formal test counted towards their course mark, but was not used as a measuring instrument. Since student results were encouraging we continued with other aspects of quantified statements.

There are many statements in mathematics that contain more than one quantifier. Hence we examined formal and informal statements that contained multiple quantifiers. We then proceeded to negations of multiple quantified statements. The following two

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generalizations were used to do a number of formal and informal examples of multiple quantified statements:

The negation of $\forall x, \exists y$ *such that* $P(x,y)$ is logically equivalent to $\exists x$ *such that* $\forall y, \sim P(x,y)$

The negation of $\exists x$ *such that* $\forall y$ $P(x,y)$ is logically equivalent to $\forall x, \exists y$ *such that* $\sim P(x,y)$

5.4.4.1 The Contra-positive, Converse and Inverse of Universal Conditional statements

Given a statement of the form: $\forall x \in D$, *if* $P(x)$ *then* $Q(x)$,
Then its **contra-positive i**s the statement: $\forall x \in D$, *if* $\sim Q(x)$ *then* $\sim P(x)$
Its **converse** is the statement : $\forall x \in D$, *if* $Q(x)$ *then* $P(x)$ and
Its **inverse** is the statement: $\forall x \in D$, *if* $\sim P(x)$ *then* $\sim Q(x)$

The following is an example where the contra-positive, converse and inverse were applied:

Example 3
Rewrite the following informal sentence in formal language and then give its contra-positive, converse and inverse:
If a real number is greater than 2, then its square is greater than 4.

Solution
Formal statement: $\forall x \in \Re$, *if* $x \rangle 2$, *then* $x^2 \rangle 4$
Contra-positive: $\forall x \in \Re$, *if* $x^2 \leq 4$ *then* $x \leq 2$ [if the square of a real number is less than or equal to 4, then the number is less than or equal to 2]
Converse: $\forall x \in \Re$, *if* $x^2 \rangle 4$, *then* $x \rangle 2$ [if the square of a real number is greater than 4, then the number is greater than 2]
Inverse: $\forall x \in \Re$, *if* $x \leq 2$ *then* $x^2 \leq 4$ [if a real number is less than or equal to 2, then the square of the number is less than or equal to 4]

It was shown previously that a conditional statement is logically equivalent to its contra-positive and that it is not logically equivalent to either its converse or its inverse. The same is true for universal conditional statements i.e.:
$\forall x \in D$, *if* $P(x)$ *then* $Q(x) \equiv \forall x \in D$, *if* $\sim Q(x)$ *then* $\sim P(x)$ and

$\forall x \in D$, *if* $P(x)$ *then* $Q(x) \not\equiv \forall x \in D$ *if* $Q(x)$ *then* $P(x)$ and

$\forall x \in D$, *if* $P(x)$ *then* $Q(x) \not\equiv \forall x \in D$, *if* $\sim P(x)$ *then* $\sim Q(x)$

The logical equivalence of a universal conditional statement to its contra-positive is a very useful tool when dealing with complex statements.

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It was shown that necessary, sufficient and only if can be extended to apply to universal conditional statements. Some examples were dealt with where this was illustrated. Extensive exercises were done where students were required to write the contra-positive, converse, inverse and negations of both formal and informal universal statements.

5.4.4.2 Universal Instantiation, Universal Modus Ponens and Universal Modus Tollens

The rule of universal instantiation states that if some property is true of everything in a domain, then it is true of any particular thing in the domain. The rule of universal instantiation can be combined with modus ponens to obtain the rule called universal modus ponens. The valid argument form of universal modus ponens can be represented as follows:

$$\forall x, \textit{ if } P(x) \textit{ then } Q(x)$$
$$P(a) \textit{ for a particular } a$$
$$\therefore Q(a)$$

The argument form of universal modus ponens consists of two premises and a conclusion and where at least one premise is quantified. The first premise is called the major premise and the second is known as the minor premise. Students were shown examples from high school where they have drawn conclusions using universal modus ponens. The application of the Pythagorean theorem is a very good illustration of this:
The Pythagorean theorem states that if you have any right-angled triangle with hypotenuse c and legs a and b, then $c^2 = a^2 + b^2$. If you are then given a particular triangle in which the legs are 3 and 4 respectively then the hypotenuse c can be evaluated as follows: $c^2 = 3^2 + 4^2 = 25$

$$\therefore c = 5$$

This was done to show students that logic is not an isolated topic, but is applied regularly at all levels in mathematics. The use of universal modus ponens in proofs were illustrated by means of a few examples.

Universal modus tollens forms an important ingredient of proof by contradiction. It has the following valid argument form:

$$\forall x, \textit{ if } P(x) \textit{ then } Q(x)$$
$$\sim Q(x) \textit{ for a particular}$$
$$\therefore \sim P(a)$$

The validity of universal modus tollens results from combining universal instantiation with modus tollens. Exercises in formal and informal language were done to help students to recognize the valid argument forms of modus ponens and modus tollens.

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5.4.4.3 Valid and invalid arguments of Quantified statements

An argument is called valid if, and only if, its form is valid. To say that an argument form is valid means that no matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting premise statements are all true, then the conclusion is also true. Specific examples of the converse and inverse error in quantified form were used to make students aware of invalid argument forms. Diagrams were utilized to test for the validity of argument forms. Fruitful discussions emanated from the use of these diagrams. Students became quite adept at using these diagrams. Examples of the use of these diagrams are the following:

Example 4
Determine whether the following arguments are valid or invalid. Support your answer by drawing diagrams:

(a) *All people are mice*
All mice are mortal
$\therefore$ *All people are mortal*

(b) *All healthy people eat an apple a day*
Helen eats an apple a day
$\therefore$ *Helen is a healthy person*

(The diagrams are shown in figures 9 and 10)

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Solution
(a)

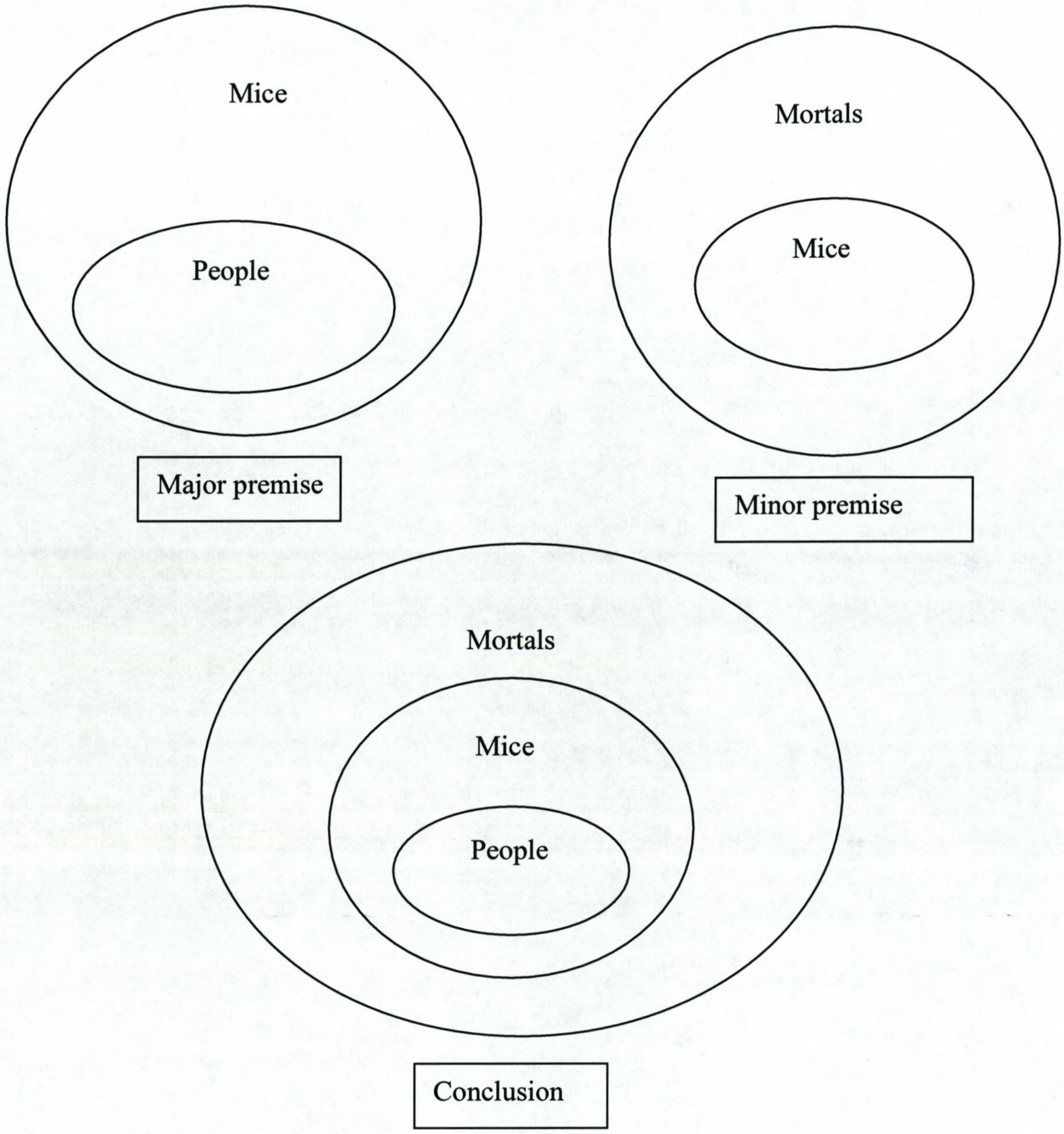

Figure 9

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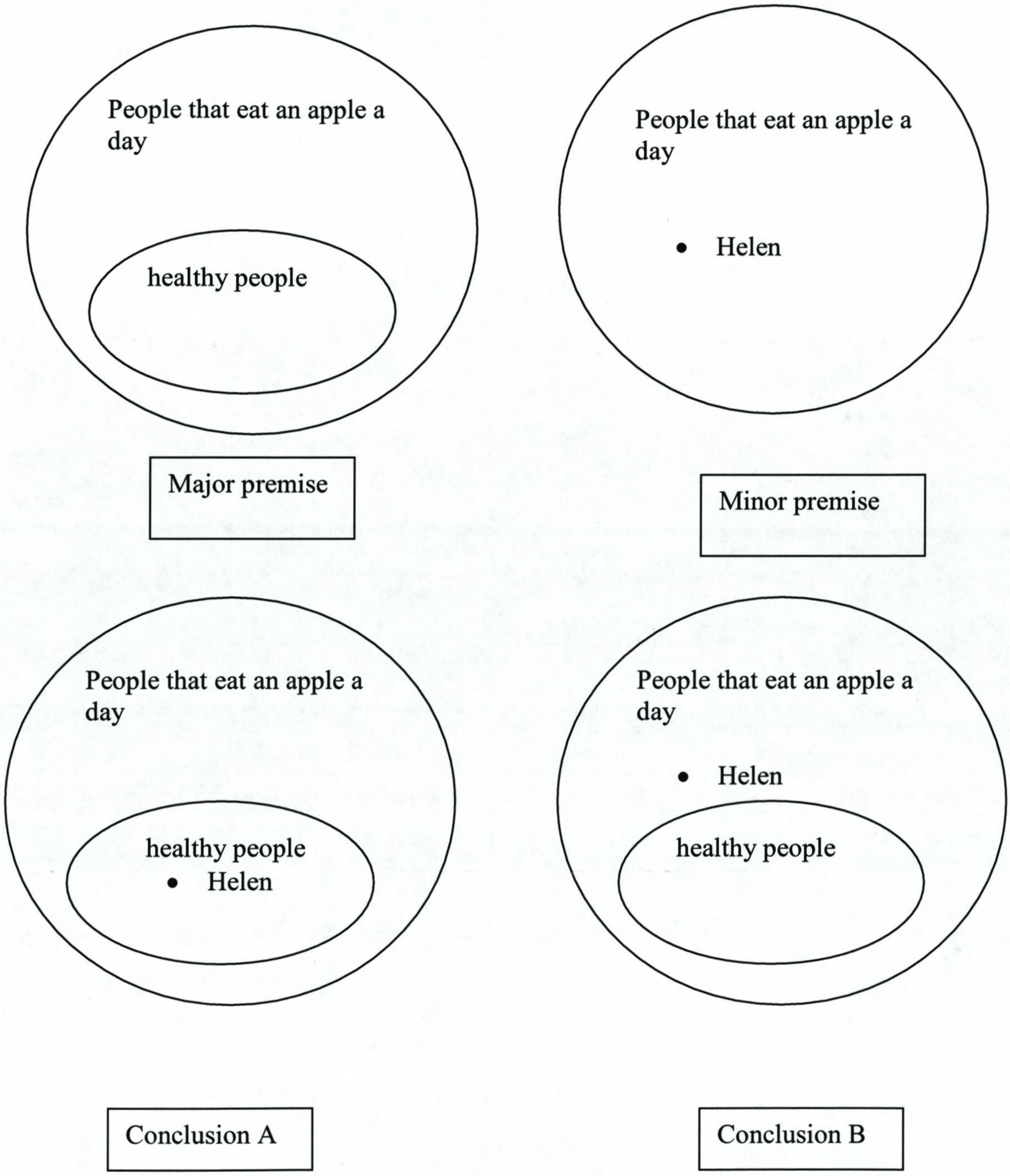
(b)



∴ Argument is invalid since the conclusion does not necessarily follow from the premises.

Figure 10

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Students got so adept at using these diagrams that they changed the minor or major premise to see if it will change the outcome. For example the following question where diagrams were used to test the validity of the argument:
No polynomial functions have horizontal asymptotes
This function has a horizontal asymptote
$\therefore$ *this function is not a polynomial*
was changed to:
no polynomial function have horizontal asymptotes
this function <u>*does not have*</u> *a horizontal asymptote*
$\therefore$ *this function is a polynomial function*

Almost the whole class participated in the discussion that ensued. Some students used set theory learnt in another class to solve the problem. They argued that there is no clear conclusion for the changed question and that their diagrams showed this. The diagram is shown below in figure 11.

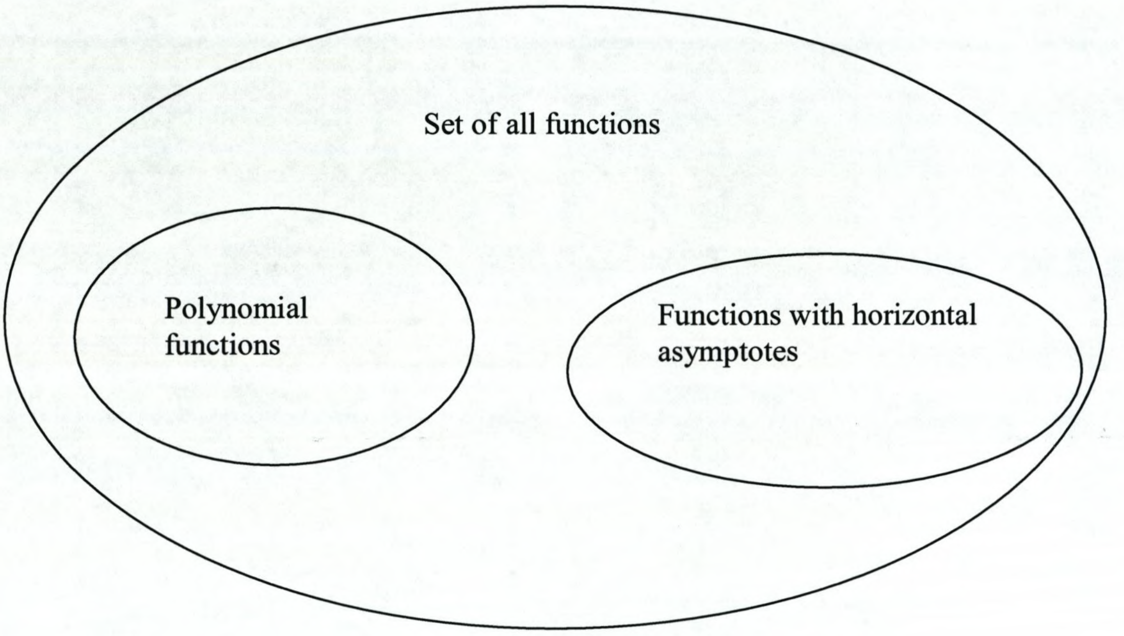

Figure 11

Based on student responses in class we were reasonably certain that they could cope with questions dealing with quantified statements. We therefore gave them the following class test on quantified statements:

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Rewrite the following in formal language:

A (i) *I trust every animal that belongs to me.*
(ii) *Dogs gnaw bones*
(iii) *I admit no animals into my study unless they will beg when told to do so.*
(iv) *All the animals in the yard are mine.*
(v) *I admit every animal that I trust into my study.*
(vi) *The only animals that are really willing to beg when told to do so are dogs.*
$\therefore$ *All the animals in the yard gnaw bones.*

B (i) *When I work a logic example without grumbling, you may be sure it is one I understand.*
(ii) *The arguments in these examples are not arranged in regular order like the ones I am used to.*
(iii) *No easy examples make my head ache.*
(iv) *I can't understand examples if the arguments are not arranged in regular order like the ones I am used to.*
(v) *I never grumble at an example unless it gives me a headache.*
$\therefore$ *These examples are not easy.*

Students did not perform very well in this test, but since the average was a pass mark we decided to continue with similar exercises. In the subsequent lecture we gave the students another test that again required them to rewrite sentences in formal language. We decided that this test would double as a pre-test[25], since we had a suspicion that all was not well where these kinds of statements were concerned. The following is this test:

Rewrite the following sentences in formal language:

A *(i)* *No birds, except ostriches, are nine feet high.*
(ii) *There are no birds in this aviary that belong to anyone but me.*
(iii) *No ostrich lives on mince pies.*
(iv) *I have no birds less than nine feet high.*

B *(i)* *All writers who understand human nature are clever.*
(ii) *No one is a true poet unless he can stir the hearts of men.*
(iii) *Shakespeare wrote Hamlet*
(iv) *No writer who does not understand human nature can stir the hearts of men.*
(v) *None but a true poet could have written Hamlet.*

To our amazement the students performed even worse in this test. The majority of students had difficulty in rewriting sentences in formal language that contained the words "*no, none, unless, never and except"* in it. We therefore decided to do intervention as the understanding of quantified statements is crucial in understanding mathematical statements. We gave them the following exercise which contained statements that contained the words that students found problematic:

[25] See Appendix C1

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Rewrite the following statements in formal language:

(i) *No easy examples in mathematics are challenging.*

(ii) *No difficult problems in mathematics can be solved easily.*

(iii) *No people except registered students are allowed to attend classes.*

(iv) *Some people will not go to church unless there is a special service.*

(v) *Some people will not buy clothes unless they have a birthday.*

The solution to the above exercise was discussed and errors were pointed out and corrected. Once we were satisfied that students could now cope with these types of sentences we gave a follow-up test[26] to determine if the erroneous reasoning was corrected. The following is this test:

Rewrite the following statements in formal language:

(i) *No bank closes before 3:30 unless it is a small bank.*

(ii) *No shark eats plankton unless it is a whale shark.*

(iii) *Students never study unless they have to prepare for a test.*

(iv) *None but a true gentleman will offer his seat to a lady on a bus.*

(v) *None but a brave soldier will fight in a war.*

There was a dramatic improvement in student performance on this follow-up test hence we were satisfied that students have now acquired the ability to deal with these kinds of statements.

Students were now presented with a post-test for Puzzle I (the pre-test for this puzzle was given in the very first lecture). The post-test is presented in Appendix A2. This puzzle is equivalent to puzzle I as only the context was changed. If we compare the puzzles we find the following similarities:

- Both puzzles had five statements
- The 1st , 2nd and 3rd statements of puzzle III contained if-then statements (compare to puzzle 1 where statements 1, 2 and 5 was if-then statements)
- Statement 3 of puzzle III contained an *or* (compare to puzzle 1 where statement 4 contained the *or*)
- Both puzzles could be proven using modus ponens, modus tollens and then disjunctive syllogism
- Both puzzles contained only compound statements.

We felt that students were also ready now to deal with forming of conclusions when presented with arguments containing quantified statements. We therefore gave students the following puzzle which served as a pre-test for arguments with quantified statements. [27] The post-test for this puzzle was given approximately a month later.[28] Again we made certain that the post-test was equivalent to the pre-test. Hence if we compare the puzzles we note the following similarities:

- Both puzzles consist of 5 quantified statements and a given conclusion

[26] See Appendix C2

[27] See Appendix D1

[28] See Appendix D2

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- In both cases students were required to rearrange the statements so that the conclusion follows logically
- In both cases 5 connections between statements needed to be made in order to arrive at the conclusion

This concluded the section on logic. We then proceeded to the section on proofs.

5.5 The proof component of the course

Our intention with this section was to determine if there was a transfer of skills as far as logic was concerned. In other words, to determine if the skills of making connections between statements and forming conclusions from arguments would translate into better deductive abilities and hence proving abilities of students. Stylianides et al **[80]** are of the opinion that modus ponens and modus tollens form the basis of the methods of proof that we utilized in our study as can be seen in the following quote: "Modus ponens is the foundation of direct proof and the proof method by mathematical induction, whereas modus tollens is the foundation of indirect proof (this includes the proof methods by contradiction and by contraposition)."

What we therefore attempted to do in our study was for students to transfer the skills learnt in the context of logic to the context of proving. Some researchers argue that this kind of transfer is not possible. The experiment of reflex theorists Woodward and Thorndike where adults who after special exercises could determine the length of short lines, but could not transfer this skill to determine the length of long lines is a classic example where transfer did not take place. A closer scrutiny of Woodward and Thorndike however reveals that they do think that transfer is possible when the transfer task and the learning task are identical. Bransford et al **[7]** argue that in this view of learning transfer, where the emphasis is on identical elements of task, there is no consideration of any learner characteristics such as where attention was directed, whether relevant principals were extrapolated etc. they are of the opinion therefore that in this view the primary emphasis is on drill and practice. Anderson et al **[1]** are of similar opinion. Modern theories of learning and transfer maintained the emphasis on practice, but they specify the kinds of practice that are important and take learner characteristics like existing knowledge and strategies into account.

The approach we used to facilitate the transfer of skills from logic to proofs was by means of a cue. Our motivation for doing so is based on the following quote from Anderson et al **[1]**: "The amount of transfer depends on where attention is directed during learning. Training on cues that signal the relevance of an available skill may deserve much more emphasis than they now typically receive in instruction". Cheng et al **[14]** is of similar opinion as can be seen from the following quote: "Training was effective only when abstract principles were coupled with examples of selection problems, which served to elucidate the mapping between abstract principles and concrete instances." Our use of cues to signal the relevance of an available skill will be demonstrated when the different methods of proof are discussed.

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5.5.1 Set theory

Since the language of set theory is used in every mathematical subject and, in particular, set identities play an important role in proving certain mathematical results, a brief introduction of elementary set theory is given. Certain set identities and set inclusions are then provided to illustrate the use of rules of inference (given as cues) and to serve as an introduction to proving in mathematics which will be dealt with in subsequent sections.

5.5.1.1 Definitions and Language of set theory

The words set and element are undefined terms of set theory just as sentence, true and false are undefined terms of logic.

If *S* is a set then $a \in S$ means that *a* is a member of *S* or an object in *S*.

$a \notin S$ means *a* is not a member of *S* or a does not belong to the set S.

We often write a set as follows: $S = \{x \in A \ / \ P(x)\}$ where this means an element *x* is in *S* if and only if *x* is in *A* and *P(x)* is true.

If *A* and *B* are two sets, *A* is called a subset of *B*, if and only if every element of *A* is also an element of *B*. This is written as $A \subseteq B$

A is not a subset of the set *B* (written $A \not\subset B$) if and only if, there is at least one element of *A* that is not an element of *B*.

A is a proper subset of *B*, if and only if every element of *A* is in *B*, but there is at least one element of *B* that is not in *A*.

5.5.1.2 Set Equality

Given sets *A* and *B, A* equals *B* ($A = B$), if and only if, every element of *A* is in *B* and every element of *B* is in *A*. In symbols: $A = B \Leftrightarrow A \subseteq B \ and \ B \subseteq A$

5.5.1.3 Operations on sets

Let *A* and *B* be subsets of a universal set *U*, then the following hold:

(a) The union of *A* and *B*, denoted $A \cup B$ is the set of all elements *x in U* such that *x* is in *A* or *x* is in *B*, symbolically: $A \cup B = \{x \in U \ / \ x \in A \ or \ x \in B\}$ or
$$x \in A \cup B \Leftrightarrow x \in A \ or \ x \in B$$

(b) The intersection of *A* and *B* denoted by $A \cap B$, is the set of all elements *x* in *U* such that *x* is in *A* and *x* is in *B*. Symbolically:
$A \cap B = \{x \in U \ / \ x \in A \ and \ x \in B\}$

(c) The difference of *B* minus *A* (or the relative complement of *A* in *B*) denote $B - A$, is the set of all elements *x* in *U* such that *x* is in *B* and *x* is not in *A*, symbolically:
$B - A = \{x \in U \ / \ x \in B \ and \ x \notin A\}$

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(d) The complement of *A* denoted A^c, is the set of all elements *x* in *U* such that *x* is not in *A*, symbolically: $A^c = \{x \in U / x \notin A\}$

5.5.1.4 Element method for proving that one set is a subset of another

Let sets *X* and *Y* be given. To prove that $X \subseteq Y$, the following must be done:

(i) Suppose that *x* is a particular but arbitrarily chosen element of *X*,
(ii) Show that *x* is an element of *Y*

5.5.1.5 Element (basic) method for proving that sets are equal

Let sets *X* and *Y* be given. To prove that *X* = *Y* the following must be done:

(i) Prove that $X \subseteq Y$
(ii) Prove that $Y \subseteq X$

5.5.1.6 Proving by Division into Cases

One of the strategies that can be utilized to prove a mathematical statement is as follows: Suppose we know that: A_1 *or* A_2 *or* A_3 *or ...* A_n is true. By definition of OR at least one of the statements A_i is true (although we might not know which one). Suppose you want to deduce a conclusion *C*. That is suppose you want to show that:
$A_1 \text{ or } A_2 \text{ or } A_3 \text{ or} \ldots A_n \rightarrow C$ Then prove all the implications:
$A_1 \rightarrow C,\ A_2 \rightarrow C,\ A_3 \rightarrow C, \ldots,\ A_n \rightarrow C$ and conclude that regardless of which statement A_i happens to be true, the truth of *C* follows.

5.5.1.7 Empty set

The unique set with no elements is called the empty set. It is denoted by the symbol ϕ.
A set with no elements is a subset of every set. Symbolically: If ϕ is a set with no elements and *A* is any set then $\phi \subseteq A$.

Uniqueness of the empty set
There is only one set with no elements.

5.5.1.8 Disjoint sets

Two sets are called disjoint if, and only if, they have no elements in common.
Symbolically: *A and B* are disjoint $\Leftrightarrow A \cap B = \phi$.

Example 5
Let $A = \{1;3;5\}$ *and* $B = \{2;4;6\}$
Then $A \cap B = \phi$
$\therefore A \cap B$ is disjoint

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Example 6
Show that A – B are B are disjoint sets.

Solution:
To show that $A - B$ *and* B are two disjoint sets we need to show that: $(A-B) \cap B = \phi$
Suppose $x \in (A-B) \cap B$, then $x \in A-B$ *and* $x \in B$ - by definition of intersection
Therefore $(x \in A \text{ and } x \notin B) \text{ and } x \in B$ - by definition of difference of two sets
Hence $x \in A \text{ and } (x \notin B \text{ and } x \in B)$ - by associative law
So in particular $x \notin B \text{ and } x \in B$
This is a contradiction,
hence our supposition is false and hence $(A-B) \cap B$ has no elements
therefore $(A-B) \cap B = \phi$ - uniqueness of empty set
it follows from the definition that $A - B$ *and* B are disjoint

Mutually Disjoint Sets
The sets A_1, A_2, …,A_n are mutually disjoint (or pairwise disjoint) if and only if, no two sets A_i and A_j with $i \neq j$ have any elements in common.
Symbolically: For all $i, j = 1, 2, \ldots, n \;\; A_i \cap A_j = \phi \text{ if } i \neq j$

Example 7
$A_1 = \{3;5\}$, $A_2 = \{1;4;6\}$, $A_3 = \{2\}$, then
$A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$, $A_2 \cap A_3 = \phi$
$\therefore A_1 \; A_2 \; A_3$ are mutually disjoint.

5.5.1.9 Partition of sets

A collection of non-empty sets $\{A_1, A_2, \ldots, A_n\}$ is a partition of a set A if, and only if:
(i) $A = A_1 \cup A_2 \cup \ldots \cup A_n$
(ii) $A_1, A_2, \ldots, A_n$ are mutually disjoint

Example 8
Let $A = \{1;2;3;4;5;6\}$, $A_1 = \{1;2\}$, $A_2 = \{3;4\}$ *and* $A_3 = \{5;6\}$. Is $\{A_1, A_2, A_3\}$ a partition of A?

Solution
(i) $A = A_1 \cup A_2 \cup A_3$
(ii) $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$, $A_2 \cap A_3 = \phi$
$\therefore \{A_1, A_2, A_3\}$ is a partition of A.

5.5.1.10 Power Sets
Given a set *A*, the power set of *A,* denoted *P(A)*, is the set of all subsets of *A*.

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Example 9

Find the power set of the set $\{x; y\}$, that is, find $P(\{x; y\})$

Solution

$P(\{x; y\}) = \{\phi, \{x\}, \{y\}, \{x; y\}\}$

5.5.1.11 (a) Example 10 (Examples of proofs of set identities and set inclusions where cues on logic are given)

Prove that for all sets A and B:

(i) $A \cap B \subseteq A \left[\begin{array}{r} make\ use\ of: p \wedge q \\ \therefore p \end{array}\right]$

(ii) $A \subseteq A \cup B \left[\begin{array}{r} make\ use\ of: p \quad \\ \therefore p \vee q \end{array}\right]$

(iii) $A - B = A - (A \cap B) \left[\begin{array}{r} make\ use\ of\ the\ rule\ of\ \inf erence: p \vee q \\ \sim p \\ \therefore q \end{array}\right]$

(iv) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \left[\begin{array}{r} make\ use\ of: p \quad \\ \therefore p \vee q \end{array}\right]$

Solutions

(i) Suppose $x \in A \cap B$. Then $x \in A$ and $x \in B$ and so in particular $x \in A$. This shows that $A \cap B \subseteq A$.

(ii) Suppose $x \in A$. Then $x \in A$ or $x \in B$ and so $x \in A \cup B$. Hence $A \subseteq A \cup B$.

(iii) Suppose $x \in A - (A \cap B)$. Then $x \in A$ and $x \notin A \cap B$
Hence $x \in A$ and (either $x \notin A$ *or* $x \notin B$)
Therefore $x \in A$ and $x \notin B$
Hence $x \in A - B$; proving that $A - (A \cap B) \subseteq A - B$
Conversely suppose $x \in A - B$, then $x \in A$ and $x \notin B$
Hence $x \in A$ and $x \notin A \cap B \left[\begin{array}{l} p \\ \therefore p \vee q \end{array}\right]$
and so $x \in A - (A \cap B)$
$\therefore \quad A - B \subseteq A - (A \cap B)$
$\therefore \quad A - B = A - (A \cap B)$

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(iv) Suppose $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$

Case 1: $x \in A$

Since $x \in A$, $x \in A \cup B$ *and* $x \in A \cup C$ $\begin{bmatrix} p \\ \therefore p \vee q \end{bmatrix}$

Hence $x \in (A \cup B) \cap (A \cup C)$ and so $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Case 2: $x \in B \cap C$

Since $x \in B \cap C$, $x \in B$ *and* $x \in C$

Hence $x \in A \cup B$ *and* $x \in A \cup C$ $\begin{bmatrix} p \\ \therefore p \vee q \end{bmatrix}$

Therefore $x \in (A \cup B) \cap (A \cup C)$

And so $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Therefore regardless of which case is true, $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Conversely: suppose $x \in (A \cup B) \cap (A \cup C)$

Then $x \in A \cup B$ *and* $x \in A \cup C$ and so

($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$)

Case 1: $x \in A$

If $x \in A$, then $x \in A \cup (B \cap C)$ $\begin{bmatrix} p \\ \therefore p \vee q \end{bmatrix}$, so

$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

Case 2: $x \notin A$

Then $x \in B$ *and* $x \in C$, so $x \in B \cap C$

And hence $x \in A \cup (B \cap C)$ $\begin{bmatrix} p \\ \therefore p \vee q \end{bmatrix}$

$\therefore (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

$\therefore A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(b) The following list of set identities can be proved in the same way as the above examples.

Let all sets referred to below be subsets of a universal set *U*.

(i) Commutative laws: for all sets *A and B*

(a) $A \cap B = B \cap A$ and (b) $A \cup B = B \cup A$

(ii) Associative laws: for all sets *A, B and C*

(a) $(A \cap B) \cap C = A \cap (B \cap C)$ and (b) $(A \cup B) \cup C = A \cup (B \cup C)$

(iii) Distributive laws: For all sets *A, B and C*

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

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(iv) Intersection with U for all sets A:
$A \cap U = A$

(v) Double Complement Law: For all sets A
$(A^c)^c = A$

(vi) Idempotent laws: For all sets A
(a) $A \cap A = A$ and (b) $A \cup A = A$

(vii) De Morgan's Laws: For all sets A *and* B
(a) $(A \cup B)^c = A^c \cap B^c$ and (b) $(A \cap B)^c = A^c \cup B^c$

(viii) Union with U: $A \cup U = U$

(ix) Absorption Laws: For all sets A *and* B
(a) $A \cup (A \cap B) = A$ and (b) $A \cap (A \cup B) = A$

(x) Alternate representation for set difference: For all sets A *and* B
$A - B = A \cap B^c$

Proving set identities using the list in 5.5.1.11 (b)

For example the statement "For all sets A, B and C, $(A - B) \cap (C - B) = (A \cap C) - B$" is proved by applying the set identities in a step by step fashion and giving a reason for each step as can be seen below.

$(A - B) \cap (C - B)$

$= (A \cap B^c) \cap (C \cap B^c)$, alternate representation for set difference.

$= (A \cap B^c) \cap (B^c \cap C\ )$, commutative law.

$= (A \cap B^c) \cap B^c) \cap C$, associative law.

$= (A \cap (B^c \cap B^c)) \cap C$, associative law.

$= (A \cap B^c) \cap C$, idempotent law.

$= A \cap (B^c \cap C)$, associative law.

$= A \cap (C \cap B^c)$, commutative law.

$= (A \cap C) \cap B^c$, associative law.

$= (A \cap C) - B$, alternate representation for set difference.

Students were given a number of exercises to consolidate the theory on sets. Once we were certain that they have acquired the necessary skills we gave them a pre-test[29] on proofs in set theory. Approximately a week later they were given an equivalent

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[29] See Appendix E1

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post-test[30]. Students also wrote a class test on proofs of set theory, this test formed part of their course mark but was not utilized as a measuring instrument in our study. The June examinations also contained questions where students were required to prove statements on sets using both the element argument and set identities.

5.5.2 Elementary Number Theory

Before continuing the discussion on different methods of proof, we give a brief outline of some of the elementary properties of numbers. This is because the examples that will be used to explain the proof methods involve number theoretic properties. We first give some definitions and examples of the different concepts.

Definitions

An integer n is even if, and only if, $n = 2k$ for some integer k.

An integer n is odd if, and only if, $n = 2k + 1$ for some integer k

An integer n is prime if, and only if $n > 1$ and for all positive integers r and s, if $n = r\ s$ then $r = 1$ *or* $s = 1$

An integer n is composite if, and only if $n = r \times s$ for some positive integers r and s with $r \neq 1$ *and* $s \neq 1$

A real number r is rational if, and only if $r = \dfrac{a}{b}$ for some integer a *and* b with $b \neq 0$

A real number is irrational if it is not rational.

Floor
Given any real number x, the floor of x denoted $\lfloor x \rfloor$, is defined as follows:
$\lfloor x \rfloor$ = that unique integer n such that $n \leq x < n+1$.

Ceiling
Given any real number x, the ceiling of x denoted $\lceil x \rceil$, is defined as follows:
$\lceil x \rceil$ = that unique integer n such that $n-1 < x \leq n$

Composite number
If n is an integer that is greater than 1, then n is composite if and only if $\exists$ positive integers r and s such that $n = r.s$ and $r \neq 1$ and $s \neq 1$

Composite number lemma
Let n be an integer that is greater than 1. If n is composite then $n = r.s$ where $1 < r < n$ *and* $1 < s < n$

[30] See Appendix E2

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Common divisor

Let *a and b* be two non-zero integers. Then there is a unique integer *d* such that:

(i) *d* is a common divisor of *a and b*, that is *d/a* and *d/b* and

(ii) if *c* is an integer which is also a common divisor, that is *c/a* and *c/b* then *c/d*.

We call *d* the greatest common divisor of *a and b*.

Example 11

Prove that 0 is an even number

Solution

$0 = 2.\ 0$

$\therefore 0$ is even by definition of even

Example 12

Prove that – 301 is odd

Solution

-301 = 2 x (-151) + 1

$\therefore -301$ is odd by definition of odd

Example 13

Is 1 *a prime number?*

Solution

1 is not a prime number, since any prime number is greater than 1

Example 14

Is it true that every integer greater than 1 *is either prime or composite?*

Solution

Yes, since the two definitions are negations of one another.

If *n* and *d* are integers and $d \neq 0$, then *n* is divisible by *d* if, and only if, *n* = *dk* for some integer *k*. The notation *d/n* is read "*d* divides *n*"

Quotient – Remainder theorem

Given any integer n and positive integer d, there exist unique integers *q* and *r* such that

$$n = d \cdot q + r \quad and \quad 0 \leq r < d$$

Example 15

n = 17; *d* = 3, then

$$17 = \underset{\substack{\downarrow \\ d}}{3} \times \underset{\substack{\downarrow \\ q}}{5} + \underset{\substack{\downarrow \\ r}}{2}$$

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5.5.3 Methods of Proof

Each method of proof follows a fixed procedure. Unlike the section on logic where we gave the students a pre-test before teaching them, we first taught the students what the procedure was for a specific method of proof and then gave the pre-test. Using the errors in reasoning of the pre-test as a guide we proceeded to teach the students again so as to eliminate these errors. Once we were satisfied that they were now competent with a specific method of proof we administered the post-test. The reason for this was because students had no previous experience of these methods of proof and as previously discussed, the students need cognitive structures that already exist for the process of assimilation and accommodation to occur.

We now proceed to illustrate the different methods of proof by examples.

5.5.3.1 The method of direct proof

The method of direct proof is based on the method of generalizing from the generic particular. In a direct proof you start with the hypothesis of a statement and make one deduction after another until you reach the conclusion. This implies the following:

To prove a statement of the form $\forall x \in D$, *if* $P(x)$ *then* $Q(x)$ you suppose x is a particular but arbitrarily chosen element of D that satisfies $P(x)$ and then you show that x satisfies $Q(x)$.

The steps involved in the method of direct proof can therefore be stated as follows:

(i) Express the statement to be proved in the form $\forall x \in D$, *if* $P(x)$ *then* $Q(x)$
(ii) Start the proof by supposing x is a particular but arbitrarily chosen element of D for which the hypothesis $P(x)$ is true
(iii) Show that the conclusion $Q(x)$ is true by using definitions, previously established results and the rules of inference

It is clear that in the case of direct proof the first two steps are easy to learn. The third step however requires the student to be creative in order to prove the conclusion. This third step therefore was our focus in the teaching process where we again emphasized step – by- step reasoning as a cognitive tool.

Example 16
Prove that the product of any two odd integers is odd

Solution
Step 1
$\forall m, n \in \mathrm{Z}$, if m and n are odd then $m.n$ is odd
Step 2
Suppose m and n are particular but arbitrarily chosen integers so that *m and n* are both odd.

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Step 3

Then by definition of odd $n = 2k + 1$ *and* $m = 2r + 1$ for some integers *k and r*
Then $m.\ n = (2k + 1)(2r + 1)$, by substitution and so $m.\ n = 4kr + 2k + 2r + 1$
$= 2(2kr + k + r) + 1$, by applying the rules of algebra. Let $s = 2kr + k + r$.
Now s is an integer because products and sums of integers are integers and *2, r and k* are all integers. Hence $m.n = 2\,s + 1$ and so by definition of odd, $m.n$ is odd.

Example 17

Prove that the sum of any two rational numbers is rational.

Solution

Step 1

$\forall$ real numbers r and s, if r and s are rational then $r + s$ is rational

Step 2

Suppose r and s are particular but arbitrarily chosen rational numbers.

Step 3

Then $r = \frac{a}{b}$ *and* $s = \frac{c}{d}$ for some integers a, b, c and d with $b \neq 0$ *and* $d \neq 0$, by definition of rational. Therefore

$$r + s = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

Then $ad + bc$ and bd are integers because products and sums of integers are integers and because a, b , c and d are all integers. Also $bd \neq 0$, by the zero product property.
Thus $r + s$ is rational by definition of rational numbers.

Example 18

Prove that for all integers *a, b* and *c* if a/b and b/c then a/c

Solution

Step 1

$\forall\ a, b, c \in \mathrm{Z}$, if a/b and b/c, then a/c

Step 2

Suppose *a, b and c* are particular but arbitrarily chosen integers such that *a/b and b/c*

Step 3

Since a/b, $b = ak$ for some integer k and since b/c, $c = bs$ for some integer s.
By substitution $c = (a.k).s = a(k.s)$, by the associative law for multiplication.
Let $p = k.s$. Then p is an integer since it is a product of integers and therefore $c = a.p$ where p is an integer and thus a divides c ,by definition of divisibility.

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Example 19 (method of division into cases)
Prove that any two consecutive integers have opposite parity.

Solution
Two integers are called consecutive if, and only if one is one more than the other. Opposite parity means one integer is odd and the other is even.

Suppose that m and $m + 1$ are two particular but arbitrarily chosen consecutive integers. By the parity property, either m is even or m is odd.

Case 1: (m is even)

In this case m $= 2k$ for some integer k, and so $m + 1 = 2k + 1$, which is odd. Hence in this case one of m and $m + 1$ are even and the other is odd.

Case 2: (m is odd)

In this case $m = 2k + 1$ for some integer k, and so
$m + 1 = (2k + 1) +1 = 2k + 2 = 2(k + 1)$, but $k + 1$ is an integer because it is a sum of two integers. Therefore $m + 1$ equals twice some integer and thus $m + 1$ is even. Hence in this case also one of m and $m + 1$ is even and the other is odd.
It follows that regardless of which case actually occurs for the particular m and $m + 1$ that are chosen, one of m and $m + 1$ is even and the other is odd

Example 20 (Application of Quotient-Remainder theorem)
Prove that the square of any odd integer has the form $8m + 1$ for some integer m.

Solution
Step 1 (formal restatement)
$\forall$ odd integers n, $\exists$ an integer m such that $n^2 = 8m + 1$

Step 2
Suppose n is a particular but arbitrarily chosen odd integer
Step 3
Then n can be written in one of the forms: $4q$ or $4q + 1$ or $4q + 2$ or $4q + 3$ for some integer q.
Now since n is odd and $4q$ and $4q + 2$ are even n must have one of the forms: $4q + 1$ or $4q + 3$

Case 1: ($n = 4q + 1$ for some integer q)
Since $n = 4q + 1$
$n^2 = (4q + 1)^2$
$n^2 = 16q^2 + 8q + 1$
$n^2 = 8(2q^2 + q) + 1$
let $m = 2q^2 + q$, then m is an integer since 2 and q are integers and sums and products are integers, so substituting:

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$n^2 = 8m + 1$ where m is an integer

Case 2: ($n = 4q + 3$ for some integer q)

Since $n = 4q + 3$

$n^2 = (4q + 3)^2$

$n^2 = 16q^2 + 24q + 9$

$n^2 = 16q^2 + 24q + (8 + 1)$

$n^2 = 8(2q^2 + 3q + 1) + 1$

let $m = 2q^2 + 3q + 1$, then m is an integer since 2, 3 and q are integers and sums and products of integers are integers. Hence substituting:

$n^2 = 8m + 1$ where m is an integer.

Case 1 and 2 show that given any odd integer, whether of the form $4q + 1$ *or* $4q + 3$, $n^2 = 8m + 1$, which is what we needed to show.

Example 21 (Floor)

Prove that for all real numbers x and for all integers m, $\lfloor x+m \rfloor = \lfloor x \rfloor + m$

Solution

Step 1

$\forall x, m \in \Re$ *if* $x \in \Re$ *and* $m \in Z$, *then* $\lfloor x+m \rfloor = \lfloor x \rfloor + m$

Step 2

Suppose x is a particular but arbitrarily chosen real number and m is a particular but arbitrarily chosen integer. Let $n = \lfloor x \rfloor$

Step 3

Then n is an integer and $n \le x < n+1$ - by definition of floor

Add m to all sides to obtain: $n+m \le x+m < n + m + 1$

Now $n + m$ is an integer since n and m are integers and a sum of integers is an integer, and so by definition of floor: $\lfloor x+m \rfloor = n + m$, but $n = \lfloor x \rfloor$, hence by substitution $\lfloor x+m \rfloor = \lfloor x \rfloor + m$, which is what was to be shown.

Example 22 (Floor)

Prove that for any integer n, $\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n}{2} & if\ n\ is\ even \\ \frac{n-1}{2} & if\ n\ is\ odd \end{cases}$

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Solution

Suppose n is a particular but arbitrarily chosen integer. By the quotient-remainder theorem, n is odd or n is even that is: $n = 2q$ or $n = 2q + 1$ for some integer q.

Case 1: $n = 2q \Rightarrow q = \dfrac{n}{2}$

Then

$$\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{2q}{2} \right\rfloor$$
$$= \lfloor q \rfloor$$
$$= q$$
$$= \frac{n}{2}$$

As was to be shown.

Case 2: $n = 2q + 1 \quad \Rightarrow \quad \dfrac{n-1}{2} = q$

Now

$$\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{2q+1}{2} \right\rfloor$$
$$= \left\lfloor q + \frac{1}{2} \right\rfloor$$
$$= q$$
$$= \frac{n-1}{2}$$

This is what was required.

Example 23 (Ceiling)

Prove that for any odd integer n, $\left\lceil \dfrac{n^2}{4} \right\rceil = \dfrac{n^2+3}{4}$

Solution

Step 1

$\forall$ odd integers n, $\left\lceil \dfrac{n^2}{4} \right\rceil = \dfrac{n^2+3}{4}$

Step 2

Suppose n is a particular but arbitrarily chosen odd integer. Then $n = 2k + 1$ for some integer k.

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Step 3
Hence

$$\left\lceil \frac{n^2}{4} \right\rceil = \frac{(2k+1)^2}{4}$$

$$= \left\lceil \frac{4k^2+4k+1}{4} \right\rceil$$

$$= \left\lceil k^2+k+\frac{1}{4} \right\rceil$$

$$\therefore \left\lceil \frac{n^2}{4} \right\rceil = k^2 + k + 1$$

also

$$\frac{n^2+3}{4} = \frac{(2k+1)^2+3}{4}$$

$$\frac{n^2+3}{4} = \frac{4k^2+4k+1+3}{4}$$

$$\frac{n^2+3}{4} = \frac{4k^2+4k+4}{4}$$

$$\frac{n^2+3}{4} = k^2+k+1$$

$\therefore$ LHS = RHS, this is what was to be shown.

Students were provided with a number of exercises where they had to prove statements using the direct method of proof.

5.5.3.2 Disproof by counterexample

To disprove a statement of the form $\forall x \in D,\ if\ P(x)\ then\ Q(x)$ find a value of x in D for which $P(x)$ is true and $Q(x)$ is false. Such an x is called a counterexample.

Example 24
Disprove the following statement by finding a counterexample:
$\forall$ real numbers a and b , if $a^2 = b^2$ then $a = b$

Solution
Let $a = 1$ *and* $b = -1$
Then $a^2 = 1^2 = 1$ and $b^2 = (-1)^2 = 1$ and so $a^2 = b^2$, but $a \neq b$ since $1 \neq -1$

Students were required to practice and hence consolidate what was learnt by doing similar exercises.

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5.5.3.3 Method of contradiction

Epp **[33]** states that argument by contradiction is based on the fact that either a statement is true or false but not both. Thus the point of departure for a proof by contradiction is the supposition that the statement to be proved is false and the goal is to reason to a contradiction. Hence as previously indicated the method of proof by contradiction consists of the following steps:

(i) Suppose the statement to be proved is false (i.e. use negation)
(ii) Show that this supposition leads logically to a contradiction
(iii) Conclude that the statement to be proved is true.

In each method of proof there is a part where the student has to apply prior knowledge of various types in order to do the proof and this is what we attempted to improve in students. In other words we tried to get them to the point where they became creative in their use of prior knowledge and hence to apply knowledge that we had not taught them. The following two examples will illustrate this.

Example 25
Prove that the sum of any rational number and any irrational number is irrational

Solution
Step 1: Suppose not [here the student is supposed to take the negation of the statement and suppose it to be true]

Suppose there is a rational number r and an irrational number s such that $r + s$ is rational.

Step 2: We must now induce a contradiction

By definition of rational $r = \dfrac{a}{b}$ *and* $r+s=\dfrac{c}{d}$ for some integers a, b , c and d with $b \neq 0$ *and* $d \neq 0$. By substitution we have:

$$\frac{a}{b} + s = \frac{c}{d} \text{ and so}$$

$$s = \frac{c}{d} - \frac{a}{b}$$

$$= \frac{bc - ad}{bd}$$

Now $bc - ad$ *and* bd are both integers and $bd \neq 0$
Hence s is a quotient of the two integers $bc - ad$ and with $bd \neq 0$
So by definition of rational s is rational
This contradicts the supposition that s is irrational.

Step 3:
Hence the supposition is false and the statement is true.

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Example 26
Prove that there is no greatest integer.

Solution
Step 1:
Suppose there is a greatest integer N

Step 2:
Since N is the greatest integer $N \geq n$ for every integer n
Let $M = N + 1$
Now M is an integer since it is a sum of integers
Also $M > N$ since $M = N + 1$
Thus M is an integer that is greater than the greatest integer, which is a contradiction.

Step 3:
This contradiction shows that the supposition is false and hence the statement is true.

Now if we compare the two examples we notice that step 1 and 3 are similar in both cases. In step 1 in both cases the negation of the given statement was assumed to be true. Step 3 in both cases stated that since a contradiction was induced the supposition was false and hence the original statement was true. It is therefore easier to teach step 1 and 3 since it follows the same pattern every time.

Step 2 requires the student to use some kind of prior knowledge. If we compare step 2 of the two examples we see that a different technique was utilized to induce a contradiction in each case (in other methods of proof it might not be step 2, but a similar argument holds). This is what we wanted to improve in our students i.e. the ability to be creative in their use of prior knowledge. We also wanted to inculcate in them the notion of one-step-reasoning. Our argument therefore is that we cannot teach the students all the different techniques that are necessary to do step 2 since there is such a vast number of mathematical statements. The best we can do is to teach them how to go about finding the specific strategy to be employed in order to find that specific technique. In other words to teach them the cognitive processes that is needed to get to the solution for the specific mathematical proof. Epp **[33]** explains this as follows: "In order to evaluate the truth or falsity of a statement one needs to understand what the statement is about. You need to know the meanings of all terms that occur in a statement since mathematicians define terms carefully and precisely."

Example 27
Prove that for all integers n, if n^2 is even then n is even.

Solution
Step 1
Suppose there exist an integer n such that n^2 is even and n is odd.

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Step 2
Suppose n is an integer such that n^2 is even but n is odd.
Now n odd implies $n = 2k + 1$ for some integer k, by definition.
[Teaching strategy - What is the next step? Look for cues in the problem. Observe that n^2 appears, so calculate it]
Hence $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2l + 1$, where l is the integer $2k^2 + 2k$. But then n^2 is odd.
So we have n^2 is even and n^2 is odd, a contradiction.

Step 3
Hence the supposition is false and the statement is true.

Example 28
Prove that $\sqrt{2}$ *is irrational*

Solution
Step 1
Suppose $\sqrt{2}$ is rational

Step 2
Then there are integers m and n with no common factors so that:

$$\sqrt{2} = \frac{m}{n} \qquad \ldots(1)$$

Squaring both sides gives:

$$2 = \frac{m^2}{n^2} \qquad \ldots(2)$$

Or equivalently: $\quad m^2 = 2n^2 \qquad \ldots(3)$

Then m^2 is even, by definition of even. It follows that m is even by example 27.

Then $\quad m = 2k$ for some integer $k \qquad \ldots(4)$
Substituting (4) into (3) gives:

$$m^2 = (2k)^2 = 4k^2 = 2n^2 \qquad \ldots(5)$$

dividing both sides of the right –most equation of (5) gives:

$$n^2 = 2k^2$$

Consequently n^2 is even and so n is even by example 27.
But we also know that m is even.
Hence both m and n have a common factor of 2, but this contradicts the supposition that m and n have no common factors.

Step 3
Hence the supposition is false and so the statement is true.

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5.5.3.4 Method of proof by contraposition

This method is based on the logical equivalence between a statement and it's contra-positive. To prove a statement by contraposition, you take the contra-positive of the statement, prove the contra-positive by a direct proof and conclude that the original statement is true. The method of proof by contraposition therefore consists of the following steps:

(i) Express the statement to be proved in the form: $\forall\, x \in D,\ \textit{if}\ P(x)\ \textit{then}\ Q(x)$

(ii) Rewrite this statement in the contra-positive form: $\forall\, x \in D,\ \textit{if}\ \sim Q(x)\ \textit{then} \sim P(x)$

(iii) Prove the contra-positive by a direct proof:

(a) Suppose x is a particular but arbitrarily chosen element of D such that $Q(x)$ is false.

(b) Show that $P(x)$ is false.

Example 29

Prove that the negative of any irrational number is irrational.

Solution

Step 1

$\forall$ real numbers x, if x is irrational then $-x$ is irrational.

Step 2

$\forall$ real numbers x, if $-x$ is not irrational then x is not irrational OR equivalently:
Since $-(-x) = x$: $\forall$ real numbers x, if x is rational then $-x$ is rational.

Step 3

(a) Suppose x is any particular but arbitrarily chosen rational number.

(b) By definition of rational $x = \dfrac{a}{b}$ for some integers a and b with $b \neq 0$

Then $-x = -\left(\dfrac{a}{b}\right) = \dfrac{-a}{b}$

Since both $-a$ *and* b are integers and $b \neq 0$, $-x$ is rational, by definition of rational, as was to be shown.

Example 30

It was mentioned earlier that the use of cues was employed as a strategy throughout our teaching on methods of proof. We show how logical equivalence of statement forms can be used in a proof by contra-position. Consider the following problem.

Use proof by contra-position to show that the difference of any rational number and any irrational number is irrational. Make use of the following logical equivalence in your proof by contraposition: $p \to q \vee s \equiv p \wedge \sim q \to s$

Solution

Here the cue was given by the logical equivalence and the idea was to get students to re-write the statement using the equivalence learnt in logic, in such a way so as to make it

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possible to prove the statement by direct proof which would otherwise be very difficult to do.

The formal statement is given by:
$\forall$ *real numbers r and s, if r is rational and s is irrational, then r – s is irrational*

The contra-position of this statement is:
$\forall$ *real numbers r and s, if r – s is rational then r is irrational or s rational*

Using the logical equivalence the statement becomes:
$\forall$ *real numbers r and s, if r – s is rational and r is rational, then s is rational*
Suppose *r* and *s* are particular but arbitrarily chosen real numbers such that *r-s* and *r* is rational

Then $r = \frac{a}{b}$ *and* $r - s = \frac{c}{d}$ for some integers *a,b,c and d*
with $b \neq 0$ *and* $d \neq 0$ - by definition of rational,
then

$$-s = \frac{c}{d} - r$$

$$s = r - \frac{c}{d}$$

$$s = \frac{a}{b} - \frac{c}{d}$$

$$s = \frac{ad - bc}{bd}$$

But $ad - bc$ are integers because *a,b,c* and *d* are integers and products and differences of integers are integers. Also $bd \neq 0$ since $b \neq 0$ *and* $d \neq 0$
Thus by definition of rational *s* is rational as was required to prove.

5.5.3.5 Connection between proof by contradiction and proof by contraposition

Epp **[33]** argues about the connection between proof by contradiction and proof by contraposition as follows:
"In a proof by contraposition the statement $\forall x \in D$, *if* $P(x)$ *then* $Q(x)$ is proved by giving a direct proof of the equivalent statement: $\forall x \in D$, *if* $\sim Q(x)$ *then* $\sim P(x)$.
To do this, you suppose that you are given an arbitrary element of *x* in *D* such that $\sim Q(x)$. You then show $\sim P(x)$.
Exactly the same sequence of steps can be used as the heart of a proof by contradiction for the given statement. The only thing that changes is the context in which the steps are written down. To rewrite the proof as a proof by contradiction you suppose there is an *x* in *D* such that *P(x)* and $\sim Q(x)$. You then follow the steps of the proof by contraposition to deduce the statement $\sim P(x)$. But $\sim P(x)$ is a contradiction to the supposition that *P(x)* and $\sim Q(x)$."

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The reverse of this is also true i.e. if you start with proof by contradiction you can "cut and paste" a portion of this proof to form the heart of proof by contraposition. We have employed this strategy i.e. start with contradiction and then cut and paste for contraposition. The following example illustrates this:

Example 31

Prove the following statement first by contradiction and then by contraposition:

For all integers a, b and c, if $a \nmid bc$ then $a \nmid b$

Solution

Proof by contradiction

Step 1

Suppose $\exists$ integers a, b and c such that $a \nmid bc$ and a/b

Step 2

Since a/b there exists an integer k such that $b = ak$,by definition of divide
Then $bc = (ak).c$
$bc = a(k.c)$, by associative law
But kc is an integer since it is a product of integers and so a/bc - by definition of divide

Thus $a \nmid bc$ and a/bc which is a contradiction

Step 3

Hence the supposition is false and the statement is true.

Proof by contraposition

Contraposition of statement: For all integers a, b and c if a/b then a/bc

Suppose a, b and c are particular but arbitrarily chosen integers such that a/b,
then $b = ak$ - by definition of divide
then $bc = (ak).c$
$bc = a(k.c)$ - by associative law
But kc is an integer since it is a product of integers and so a/bc - by definition of divide

,as was to be shown.

The above example shows how parts of the proof by contradiction can be "cut and paste" to form the heart of proof by contraposition. [*The part in the border represents the part that was cut and pasted]*

In order to consolidate the above method of proof exercises were given that contained similar examples. Students completed pre-tests and post-tests on direct proof based on divisibility[31] and number theory[32] respectively. Although we also did pre- and post-tests for the proof by contradiction and contraposition we could not include it in our study because of an error in the statement of the problem. The students also wrote a class test

[31] See Appendices F1 and F2.

[32] See Appendices G1 and G2.

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on direct and indirect proof. Upon the completion of the section on direct and indirect proof we proceeded to the principal of mathematical induction.

5.5.4 Mathematical Induction

Mathematical induction is used to check conjectures about the outcomes of processes that occur repeatedly and according to definite patterns. The proof of a statement by mathematical induction consists of two steps:

(i) In step 1 (basis step) you prove that *P(a)* is true for a particular integer *a*.

(ii) In step 2 (inductive step) you prove that for all integers $k \geq a$, if *P(k)* is true then *P(k + 1)* is true. To prove step 2 we do the following:

(a) Suppose that *P(k)* is true where *k* is a particular but arbitrarily chosen integer $\geq a$. This supposition that *P(k)* is true , is called the inductive hypothesis.

(b) Then we prove that *P(k + 1)* is true using the inductive hypothesis. Finally we can conclude that *P(n)* is true for all integers $n \geq a$. The principal of mathematical induction can be formally represented as follows:

Let *P (n)* be a predicate that is defined for integers *n*, and let *a* be a fixed integer. Suppose the following two statements are true:

(i) *P(a)* is true

(ii) For all integers $k \geq a$, if *P (k)* is true then *P (k + 1)* is true.

Then the statement for all integers $n \geq a$, *P(n)* is true.

Example 32

Use mathematical induction to prove that: $1+2+...n = \dfrac{n(n+1)}{2}$ *for all integers* $n \geq 1$

Solution

To start one must first identify *P (n)*. In this case *P (n):* $1+2+...n = \dfrac{n(n+1)}{2}$

Basis step [We must show *P*(1) is true]

LHS: $P(1) = 1$

RHS: $P(1) = \dfrac{1(1+1)}{2} = \dfrac{2}{2} = 1$

$\therefore$ LHS = RHS

And so the formula is true for $n = 1$.

Inductive step

(a) Suppose that *k* is a particular but arbitrarily chosen integer greater than or equal to 1 such that *P(k)* is true: $P(k): 1+2+...+k = \dfrac{k(k+1)}{2}$

(b) Next we must prove *P (k+1)* is true, that is: $1+2+3+...+k+(k+1) = \dfrac{(k+1)(k+2)}{2}$

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Now $P(k+1) = 1 + 2 + 3 + \ldots + k + (k+1)$

$$= \frac{k(k+1)}{2} + k + 1$$
$$= \frac{k+1}{2}(k+2)$$
$$= \frac{k+1}{2}[(k+1)+1]$$

This proves that $P(k+1)$ is true.

$\therefore$ We conclude therefore that $1+2+\ldots n = \dfrac{n(n+1)}{2}$

Example 33

Use mathematical induction to prove the following statement:

$$1^3 + 2^3 + \ldots + n^3 = \left[\frac{n}{2}(n+1)\right]^2$$

Solution

Basis step

P(1): LHS = $1^3 = 1$

$$\text{RHS} = \left[\frac{1(1+1)}{2}\right]^2 = \left[\frac{2}{2}\right]^2 = (1)^2 = 1$$

$\therefore$ LHS = RHS

$\therefore$ formula is true for $n = 1$

Inductive step

(a) Suppose k is a particular but arbitrarily chosen integer greater than or equal to 1, such that $P(k)$ is true: $P(k): 1^3 + 2^3 + \ldots + k^3 = \left[\frac{k}{2}(k+1)\right]^2$

(b) Now we need to prove it is true for $P(k+1)$:

$$P(k): 1^3 + 2^3 + \ldots + k^3 + (k+1)^3$$
$$= \left[\frac{k}{2}(k+1)\right]^2 + (k+1)^3$$
$$= \frac{(k+1)^2}{4}\left[k^2 + 4(k+1)\right]$$
$$= \frac{(k+1)^2}{4}\left[k^2 + 4k + 4\right]$$
$$= \frac{(k+1)^2}{4}\left[(k+2)^2\right]$$
$$= \left[\frac{(k+1)(k+2)}{2}\right]^2$$

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This prove that $P(k+1)$ is true.

We conclude therefore that: $1^3+2^3+...+n^3=\left[\frac{n}{2}(n+1)\right]^2 \ \forall n\geq 1$

5.5.4.1 Using mathematical induction to prove a divisibility property

Example 34
Prove by mathematical induction that for all integers $n\geq 1$, $2^{2n}-1$ is divisible by 3.

Solution
Basis step
$P(1)$: $2^{2(1)}-1=4-1=3$, which is divisible by 3.
$\therefore P(1)$ is true.

Inductive step

(a) Inductive hypothesis
Suppose that for any particular but arbitrarily chosen integer ≥ 1, $P(k)$ is true i.e. $2^{2k}-1$ is divisible by 3.

(b) We now need to show that $P(k+1)$ is true. That is we need to show that $2^{2(k+1)}-1$ *i.e.* $2^{2k+2}-1$ is divisible by 3.

Now

$$\begin{aligned} 2^{2(k+1)}-1 &= 2^{2k+2}-1 \\ &= 2^{2k}.2^2-1 \\ &= 4.2^{2k}-1 \\ &= 4.(2^{2k}-1)+3 \end{aligned}$$

Now since $2^{2k}-1$ is divisible by 3 by the inductive hypothesis let $2^{2k}-1=3l$ for some integer l, then

$$\begin{aligned} 2^{2(k+1)} &= 4\cdot 3l+3 \\ &= 3(4l+1) \end{aligned}$$

$$\therefore 2^{2(k+1)}-1 \ \textit{is divisible by } 3$$

$\therefore P(k+1)$ is true.
Therefore $2^{2n}-1$ is divisible by 3 $\forall\, n\geq 1$

5.5.4.2 Using mathematical induction to prove an inequality

Example 35
Use mathematical induction to prove that $n^2<2^n \ \forall n\geq 5$

Solution
Basis step
P (5): LHS $=5^2=25$
RHS $=2^5=32$. And so $25<32$ $\therefore$ *P (5)* is true

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Inductive step

(a) Suppose that k is a particular but arbitrarily chosen integer ≥ 5 such that *P (k)* is true. That is $k^2 < 2^k$.

(b) We now need to prove that *P (k+1)* is true. That is $(k+1)^2 < 2^{k+1}$

Now

$(k+1)^2 = k^2 + 2k + 1 < 2^k + 2k + 1$

$(k+1)^2 < 2^k + 2^k = 2.2^k = 2^{k+1}$

$\therefore (k+1)^2 < 2^{k+1}$

$\therefore P(n) \textit{ is true } \forall n \geq 5$

The section on ordinary mathematical induction was concluded with vigorous discussion on the strategies involved in proving by mathematical induction. To consolidate exercises were given and immediate feedback on the exercises was provided. A pre- and post-test on the method of ordinary induction was administered.[33]

5.5.4.3 Strong Mathematical Induction

The lesson was started with a definition of recursively defined sequences, since at this point students did not yet deal with recursive sequences.

Definition

A sequence $a_0, a_1, a_2, \ldots$ is said to be given recursively if the first few terms are specified and a rule (called a recursion) is given for computing each later term from the earlier ones.

5.5.4.4 Teaching to connect ordinary mathematical induction to strong induction

Instead of starting the lecture in the normal way by discussing the steps involved in proving by strong mathematical induction, the lecturer used an innovative teaching strategy to link ordinary mathematical induction with strong mathematical induction. This strategy is described in the next example. The lecturer started the example and stopped where it is required to prove that the general formula represents the recursive formula.

Example 36

Suppose that $a_0, a_1, a_2, \ldots$ is the sequence such that $a_0 = 1,\ \ a_1 = 2$ and

$a_{n+2} = 2a_n + a_{n+1} \quad \forall\, n \geq 0$

Then $a_0 = 1,\ \ a_1 = 2$ and

$a_2 = 2a_0 + a_1 = 2(1) + 2 = 4 = 2^2$

[33] See Appendices H1 and H2.

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$a_3 = 2a_1 + a_2 = 2(2) + 4 = 8 = 2^3$

$a_4 = 2a_2 + a_3 = 2(4) + 8 = 16 = 2^4$

$a_5 = 2a_3 + a_4 = 2(8) + 16 = 32 = 2^5$

Based on the fact that each term can be written as 2 to a power we conjecture that the general formula of the given sequence is: $a_n = 2^n \quad \forall n \geq 0$

At this point the lecturer stopped writing on the board and pointed out that mathematical induction can be used to prove that the general formula represents the sequence. After asking pertinent questions as to how one should go about proving this conjecture it became clear that the majority of students in the class did not even know how to start such a proof. Some of the reasons advanced for this are that in their previous experience with ordinary mathematical induction everything that needed to be proved appeared in one line and also that they had previously only dealt with one initial value. In this case however $a_n = 2^n \quad \forall n \geq 0$ needs to be proved using $a_{n+2} = 2a_n + a_{n+1} \quad \forall n \geq 0$ which appeared on different lines. Most students in the class therefore could not even start, since they wanted to use only $a_n = 2^n \quad \forall n \geq 0$. Subsequently the lecturer called Mc Clean (one of the better students) to attempt a proof for the conjecture. The following is Mc Clean's proof:

From hypothesis: $\quad a_k = 2^k \textit{ where } k \geq 0$

Inductive step:

$$
\begin{aligned}
a_{k+1} &= a_{(k-1)+2} \\
&= 2a_{(k-1)} + a_{(k-1)+1} \\
&= 2a_{k-1} + a_k \\
&= 2.2^{k-1} + 2^k \qquad - \textit{from hypothesis} \\
&= 2^k + 2^k \\
&= 2.2^k \\
&= 2^{k+1}
\end{aligned}
$$

Mc Clean's use of $(k\text{-}1) + 2$ shows that he tried to get the subscript in the same form as the subscript of the a_{n+2} in $a_{n+2} = 2a_n + a_{n+1}$ (k-1 represents n). This is a clear indication that he tried to get it to a form that fits the formula, so that he could use it to expand his hypothesis.

On completion of his proof the lecturer asked Mc Clean to remain at the board. The lecturer then asked the rest of the class to point out a fundamental flaw in Mc Clean's reasoning. No one in the class could point this out. However after a few wild goose

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chases by the rest of the students Mc Clean himself started circling with his finger on the board around the part he suspected was the problem. The following is that part:

$$\begin{aligned} a_{k+1} &= a_{(k-1)+2} \\ &= 2a_{(k-1)} + a_{(k-1)+1} \\ &= 2a_{k-1} + a_k \\ &= 2.2^{k-1} + 2^k \qquad - \textit{from hypothesis} \end{aligned}$$

This indicated that he started to realize that assuming $a_k = 2^k$ which then implies $a_{k-1} = 2^{k-1}$ is where the problem lies. It also perhaps showed his dawning realization that he started with what he needed to prove i.e. that if the statement holds for *k* then it holds for all other integers. The lecturer confirmed his suspicion by stating that one cannot start with what you are supposed to prove.

Mc Clean as well as the rest of the students also did not consider the consequences of letting *k* start at 0. That is in the inductive step where $a_{k+1} = 2a_{k-1} + a_k$ if $k = 0$ then a_{-1} would have to be considered. As a result of prompting and incisive questioning from the lecturer this problem was exposed. The lecturer then suggested to the class to change this to $k \geq 1$. This was completely new to the class, the fact that this restriction could be changed, but strangely no student questioned this.
It was by now clear to the students that the ordinary method of proof by induction can not be used to prove the conjecture. The lecturer then explained that a consequence of this was that a different and stronger form of mathematical induction was required for such cases.
The above lesson was deliberately started with the given example and in the indicated manner so that students could see and identify the limitations of ordinary induction. Subsequently the students were introduced to the principle of strong mathematical induction.

5.5.4.5 Principle of Strong Mathematical Induction

Let *P (n)* be a predicate that is defined for integers *n*, and let *a* and *b* be fixed integers with $a \leq b$. Suppose the following two statements are true:

(i) *P(a), P(a+*1*),*...and *P(b)* are all true (basis step)

(ii) For any integer $k > b$, if *P* (*i*) is true for all integers *i* with $a \leq i < k$ (Inductive hypothesis). Then *P (k)* is true (inductive step)

Then the statement "for all integers, $n \geq a$ *P (n)*" is true.

Example 36 (corrected)

Take P(n) to be "$a_n = 2^n$"

Choose a = 0, b = 1

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Basis step

P(0): $a_0 = 2^0 = 1$

P(1): $a_1 = 2^1 = 2$

$\therefore$ P(0) and P(1) are true.

Inductive hypothesis

Let *k* be an integer with $k > 1$

Suppose that for all integers *i* with $0 \le i < k$, $a_i = 2^i$

Inductive step [show that *P(k)* is true, that is $a_k = 2^k$]

Now $a_k = 2a_{k-2} + a_{k-1}$, since $0 \le k-2, k-1 < k$

It follows that $a_{k-1} = 2^{k-1}$ *and* $a_{k-2} = 2^{k-2}$

Hence

$$\begin{aligned} a_k &= 2\left(2^{k-2}\right) + 2^{k-1} \\ &= 2^{k-1} + 2^{k-1} \\ &= 2.2^{k-1} \\ \therefore a_k &= 2^k \end{aligned}$$

This shows that *P(k)* is true .

So we conclude that $a_n = 2^n \;\; \forall\, n \ge 0$

5.5.4.6 Proving a property of a sequence using strong mathematical induction

Example 38

Define a sequence $a_1, a_2, a_3, \ldots$ *as follows*:

$$a_1 = 0; \quad a_2 = 2$$

$$a_k = 3a_{\left\lfloor \frac{k}{2} \right\rfloor} + 2 \quad \forall \text{ integers } k \ge 3$$

(a) *Find the first seven terms of the sequence*

(b) *Prove that* a_n *is even for each integer* $n \ge 1$

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Solution

(a)

$$a_1 = 0$$
$$a_2 = 2$$
$$a_3 = 3.a_{\left\lfloor \frac{3}{2} \right\rfloor} + 2 = 3.a_1 + 2 = 3(0) + 2 = 2$$
$$a_4 = 3.a_{\left\lfloor \frac{4}{2} \right\rfloor} + 2 = 3.a_2 + 2 = 3(2) + 2 = 8$$
$$a_5 = 3.a_{\left\lfloor \frac{5}{2} \right\rfloor} + 2 = 3.a_2 + 2 = 3(2) + 2 = 8$$
$$a_6 = 3.a_{\left\lfloor \frac{6}{2} \right\rfloor} + 2 = 3.a_3 + 2 = 3(2) + 2 = 8$$
$$a_7 = 3.a_{\left\lfloor \frac{7}{2} \right\rfloor} + 2 = 3.a_3 + 2 = 3(2) + 2 = 8$$

(b) Let *P(n)*: a_n is even

Choose $a = 1,\ b = 2$

Basis step

P(a): $a_1 = 0$ which is even

$\therefore$ *P(a)* is true.

P(b): $a_2 = 2$, which is even

$\therefore$ *P(b)* is true.

Inductive hypothesis

Let *k* be an integer with $k > 2$ and suppose that a_i is even for all integers *i* with $1 \le i < k$

Inductive step

We know $a_k = 3.a_{\left\lfloor \frac{k}{2} \right\rfloor} + 2$ *and* $k \ge 3$

Remember: $\left\lfloor \frac{k}{2} \right\rfloor = \begin{cases} \frac{k}{2} & \text{if } k \text{ is even} \\ \frac{k-1}{2} & \text{if } k \text{ is odd} \end{cases}$

And $1 \le \frac{k}{2} < k$ *and* $1 \le \frac{k-1}{2} < k$, therefore $1 \le \left\lfloor \frac{k}{2} \right\rfloor < k$

From the induction hypothesis: $a_{\left\lfloor \frac{k}{2} \right\rfloor}$ is even

So $3.a_{\left\lfloor \frac{k}{2} \right\rfloor}$ is even since $[odd \times even = even]$

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And so $a_k = 3.a_{\left\lfloor \frac{k}{2} \right\rfloor} + 2$ is also even [since it is the sum of two even numbers]

This shows that *P(k)* is true
We conclude therefore that a_n is even for all integers $n \geq 1$

After the above examples were discussed and analyzed the following similar exercise was given to students.

Example 39

Suppose $b_1, b_2, b_3, \ldots$ *is a sequence defined as follows*:

$b_1 = 3,\ b_2 = 6, \quad b_k = b_{k-2} + b_{k-1}\ \forall \text{ integers } k \geq 3$

Prove that $3 / b_n\ \forall \text{ integers } n \geq 1$

The following is the solution of Nondumo Masixolo a member of the experimental group of students:

Solution

Take *P (n)* to be the statement $3 / b_n\ \forall \text{ integers } n \geq 1$
Choose $a = 1$ and $b = 2$

Basis step

$P(1)$: $b_1 = 3\ [3 \times 1 = 3]$
$P(2)$: $b_2 = 6\ [3 \times 2 = 6]$
$\therefore P(1)$ and $P(2)$ are true

Inductive hypothesis

Let k be any integer with $k \geq 2$ and suppose $\forall \text{ integers } i \text{ with } 1 \leq i < k, \quad 3 / b_i$ is true

Inductive step

$b_k = b_{k-2} + b_{k-1}\ \forall \text{ integers } k \geq 3$
Now $k \geq 3 = 2 + 1, \ \therefore k - 2 \geq 1 \text{ and } k - 1 \geq 2$
Since $1 \leq k-2, k-1 < k$ it follows by inductive hypothesis that $3 / b_{k-1} \text{ and } 3 / b_{k-2}$
Then

$b_{k-1} = 3l \text{ for some integer } l$

$b_{k-2} = 3v \text{ for some integer } v$

$$\therefore b_k = 3l + 3v = 3(l + v)$$

Since $l+v$ is an integer $b_k = 3 \times \text{some integer}$ and $\therefore 3 / b_k$

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Discussion of Masixolo's solution

The part of the solution $k \geq 3 = 2+1,\ \therefore k-2 \geq 1\ and\ k-1 \geq 2$ shows that the student knew that what is required is to show that the subscripts *k-2* and *k*-1 is between 1 and *k* in order to use his inductive hypothesis. That is he needs to show that the formula holds for *k*-1 and *k-2* in order to prove that it holds for *n* = *k*. This in turn is an indication that he knew that in order to prove it for b_k he needed to show that $3/b_{k-1}\ and\ 3/b_{k-2}$.

He was also aware of the fact that one can prove 3 divides an integer by showing that it is $3 \times some\ integer$. The conclusion one can draw from the above solution then is that the student is cognizant of the requirements of such a proof.

Example 40

Prove that any integer greater than 1 *is divisible by a prime number.*

Solution

Let *P (n)* be the divisibility property i.e. *P (n): n* is divisible by a prime number.
Choose *a = 2 = b*

Basis step

P (a): 2/2 and 2 is prime
Therefore the divisibility property holds for *n = 2*

Inductive hypothesis

Let *k* be an integer with $k > 2$ and suppose that *i* is divisible by a prime number for all integers $2 \leq i < k$

Inductive step (proof that *k* is divisible by a prime)

Now if *k* is prime then *k/k* and so *k* is divisible by a prime number, namely *k* itself.
If *k* is composite, then *k* = *r.s* where $1 < r < k\ \ and\ \ 1 < s < k$ by the composite number lemma.
By the inductive hypothesis, $\exists$ a prime *p* such that *p/r* and since *k = r.s*, it follows that *r/k.*
Now we have *p/r* and *r/k* and so *p/k* by the transitivity of divisibility.
$\therefore$ Regardless of which case holds, there is always a prime number which divides *k*.
We conclude that every integer >1 is divisible by a prime number.

On completion of this section students were exposed to similar exercises. With some of these exercises the lecturer asked selected students to explain their solutions to the class. This was done to determine their level of understanding and also to eliminate reasoning errors. A pre- and post-test on strong mathematical induction[34] was administered. Both of these tests dealt with recursive sequences. The pre-test was administered immediately after the lesson on strong mathematical induction that dealt with recursive sequences was completed, whereas the post-test was done after reasoning errors were addressed.

[34] See Appendices I1 and I2.

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Although other topics were dealt with besides the ones stated our study was concluded with the above topic.

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CHAPTER 6

RESULTS OF THE STUDY [PRESENTATION AND DISCUSSION]

6.1 Introduction

In the following section we will present the different assessment instruments and memoranda and then discuss and analyze our data and findings. We will look at logic puzzles and proofs separately. For both logic puzzles and proofs we will analyze the pre-test first and immediately after that the post-test for the same instrument will be analyzed.

6.2 Logic puzzles

6.2.1 Puzzle I (pre – test)

Compound and conditional statements
In the back of an old cupboard you discover a note signed by a pirate famous for his bizarre sense of humour and love of logical puzzles. In the note he wrote that he had hidden treasure somewhere on the property. He listed five true statements (a – e below) and challenged the reader to use them to figure out the location of the treasure.

a) If this house is next to a lake, then the treasure is not in the kitchen.
b) If the tree in the front yard is an elm, then the treasure is in the kitchen.
c) This house is next to a lake.
d) The tree in the front yard is an elm or the treasure is buried under the flagpole.
e) If the tree in the backyard is an oak, then the treasure is in the garage.

Where is the treasure hidden?

The solution to the puzzle consists of three steps namely:

1. Using statement (a) and (c) we can conclude that **the treasure is not in the kitchen. – modus ponens (this is also known as affirming the consequent)**
2. Using statement (b) and the conclusion from 1 we can conclude that **the tree in the front yard is not an elm. – modus tollens (denying the antecedent)**
3. From (d) and the conclusion of 2 we get the final conclusion that **the treasure is buried under the flagpole**. **– disjunctive syllogism**

So in order to solve the puzzle the student had to make three connections between and from statements.

A memorandum consisting of the above solution was used to assess the students' answers.

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6.2.1.1 Analysis of assessment of puzzle I of Experimental Group

26 Students completed the puzzle. Thirteen students (50% of the total number of students) came to the correct conclusion. However 5 of these students were repeating the course, so effectively only 8 came to the correct conclusion. Six of these students (23% of the total number of students) gave a complete and correct argument that supported their conclusion. The other 2 gave no supporting arguments for their conclusion. For example one stated "*The oak normally grows next to an ocean, so it can't be* ***on*** *the garage*". This statement cannot be deduced from any of the given statements. So it appears as if the student tried to eliminate some of the possibilities by providing some motivation for doing so, although in this case the motivation did not make sense.

Thirteen students (50% of the total number of students) came to the wrong conclusion. Of these students one had no conclusion at all. Four of them made the connection that since the house is next to a lake it implies that the treasure is not in the kitchen. This shows that they have made the connection between statements (a) and (c). They however did not use this conclusion (the treasure is not in the kitchen) to make the other two connections and therefore came to the wrong conclusion. An example of this can be seen in the following student's answer "*The treasure is in the garage. Firstly this house is next to the lake so the treasure is not in the kitchen. The treasure cannot be buried under the flagpole. So it's in the garage. Statement (c) says the house is next to a lake and from statement (a) that means the treasure is not in the kitchen.*"

Eight of these students (representing 31% of the total number of students) gave muddled reasoning (muddled reasoning - meaning that there was no structure to their argument i.e. they did not show connections between facts or from facts) and came to conclusions that cannot be derived from the given statements, for example Andiswa Qosho states "*It is said that if the tree in the backyard is an oak tree then the treasure is in the garage because a treasure cannot be inside the house if the tree in the back of the yard is an oak.*" Zinnia Williams came to the conclusion that the treasure is not on the property. The reason advanced for this conclusion is "Neither *is the treasure in the front and backyard, cause there can either be an elm or an oak tree*" Siphokazi Ncwaiba stated that since statements a, b and e use the word ***if*** and since *if* is a keyword one cannot be certain if the treasure is in the kitchen, garage or under the flagpole. She goes on to state that statements c and d are more conclusive since it states what ***is*** the case. After all this reasoning the student comes to the conclusion that the treasure is either in the kitchen or under the flagpole. Thulani Shabangu's arguments are also very muddled. He states *"The treasure is hidden in a house next to a lake as stated in (c) but it cannot be in the kitchen according to clue (a). It is buried under a flagpole in a garage"*
A summary of the argument analysis of puzzle I for the experimental group is presented in table 8.

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Table 8: Argument analysis of experimental group (pre-test)

No. of connections	No. of students
0	7
1	10
2	2
3	2

6.2.1.2 Analysis of assessment of puzzle I of Control Group

50 Students participated in the study.
24 students (48% of the total number of students) had a correct answer, whilst 26 (representing 52% of students) had incorrect answers.

The 24 correct answers:
Of the 24 correct answers 8 made 0 connections between statements.
8 made one connection between statements.
5 made 2 connections between statements.
3 made 3 connections i.e. made all the required connections i.e.

The 26 Incorrect answers:
Of the 26 incorrect answers 19 made 0 connections.
5 made 1 connection between statements.
2 made 2 connections between statements.
None made 3 connections between statements i.e. there were no students in this group that made all the connections.
A summary of the argument analysis of puzzle I for the control group is presented in table 9.

Table 9: Argument analysis of puzzle I of control group (pre-test)

No. of connections	No. of students
0	27
1	13
2	7
3	3

6.2.1.3 Conclusions based on puzzle I

Based on the above analysis I think it is safe to assume that at this point in time the majority of students do not have the skill to make logical connections between statements. This skill of making connections is also required to make connections between mathematical statements when proving theorems or statements. In other words the students need to be able to do deductive reasoning. So in order to teach them to do deductive reasoning and hence reasoning abstractly we decided to teach them a course in logic. Epp states that in order for students to think abstractly, it will require them to learn to use logically valid forms of argument, to avoid common logical errors, to understand what it means to reason from definitions, and to know how to use both direct and indirect

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argument to derive new results from those already known to be true. Since proving forms such a vital part of mathematics it is crucial that students develop this skill. Hence we used these puzzles to develop in students the skill of making logical connections between and from statements.

6.2.2 Puzzle III (post test)

Your grandfather who is known for his sense of humour and love of logical puzzles left you a note. In the note he wrote that he had hidden your birthday present somewhere on one of his properties. He listed five true statements (a – e below) and challenged you to figure out the location of the present.

a) If this house is next to a main road, then the present is not in the attic.
b) If there is a swing in the yard then the present is in the study.
c) The yard has a lawn or the present is in the cupboard next to the stove.
d) If the yard has a lawn, then the present is in the attic.
e) The house is next to a main road.

Where is the present hidden?

This puzzle is equivalent to puzzle I which was used as a pre – test. This puzzle therefore was essentially the same as puzzle I only the context was changed i.e.

1. Both puzzles had 5 statements
2. The 1st , 2nd and 3rd statements of puzzle III contained if – then statements (conditional statements) (compare to puzzle I where statements 1, 2 and 5 is if – then statements)
3. Statement 3 of puzzle III was a statement containing OR (compare to puzzle I where statement 4 contained the OR)
4. Puzzle I and III could be proven using:
   (i) modus ponens
   (ii) modus tollens and then
   (iii) disjunctive syllogism
5. Both puzzles contained only compound statements.

The solution to the puzzle is as follows:

1. Using statements (a) and (e) the conclusion is that the present is not in the attic – modus ponens
2. Using (d) and the conclusion of 1 we get that the yard does not have a lawn – modus tollens
3. From (c) and the conclusion of 2 we get the final conclusion that the present is in the cupboard next to the stove – disjunctive syllogism.

So in order to solve the puzzle the student had to make three connections between and from statements.

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Alternatively the rules of inference can be used to solve the puzzle:

Using variables with valid argument forms with the rules of inference:
Let: p = the house is next to a main road
q = the present is not in the attic
r = there is a swing in the yard
s = the present is in the study
t = the yard has a lawn
u = the present is in the cupboard next to the stove

The statements *a* to *e* then become:

(a) $p \Rightarrow q$

(b) $r \Rightarrow s$

(c) $t \vee u$

(d) $t \Rightarrow \sim q$

(e) p

Solution:

1. $p \Rightarrow q$ - from (a)
p - from (e)
$\therefore q$ - modus ponens

2. $t \Rightarrow \sim q$ - from (d)
q - from (1)
$\therefore \sim t$ - modus tollens

3. $t \vee u$ - from (c)
$\sim t$ - from (2)
$\therefore u$ - disjunctive syllogism

$\therefore$ *the present is in the cupboard next to the stove*

A marking memorandum consisting of the above solutions was used to assess the students' answers.
This puzzle was given 6 weeks after the first puzzle.

6.2.2.1 Analysis of assessment of puzzle III (post – test) of experimental group

26 Students attempted the puzzle. Two of the 26 students were repeating the course and therefore will not be taken into consideration.

Two of the 24 students came to the incorrect conclusion and 22 (92% of the total number of students) came to the correct conclusion. However one of these students started midway through the first term and therefore was not present when compound statements was done in class. The other student, Andiswa Qosho, tried to use variables with the rules of inference, but only succeeded in rewriting the statements in symbol form. She could

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not connect the arguments using the rules of inference. So once again she gave a muddled argument.

Seventeen of the students used variables and the rules of inference with valid statement forms to get to the result. One of these students came to the correct conclusion but used the wrong rule of inference. Another student made a mistake in assigning the variables and therefore had to make use of more than three steps to get to the conclusion.

Four of the students used the statements without variables, but still came to the correct conclusion. One of these four students came to the correct conclusion although part of his reasoning was based on an erroneous deduction. He states: "The *yard has a lawn but this house is also next to a main road, therefore the present can't be in the attic.* ∴ *The present is in the cupboard next to the stove.*"

Of the four students that initially could not progress beyond the first connection (with puzzle I) three now got it completely correct whilst the fourth one did not do puzzle III. Three of the eight that gave muddled arguments with puzzle I, now gave a complete argument and came to the correct conclusion. Of this eight four did not do puzzle III and one ceased her studies.
A summary of the argument analysis of puzzle III for the experimental group is presented in table 10.

Table 10: Argument analysis of puzzle III of experimental group (post-test)

No. of connections	**No. of students**
0	1
1	1
2	5
3	17

6.2.2.2 Analysis of assessment of puzzle III of control group

49 Students participated in the study.
31 of the 49 had correct answers (this represents 63% of the total number of students)
18 of the 49 had incorrect answers (37% of the total number of students)

The 31 correct answers:
3 made 0 connections between statements
14 made 1 connection between statements
5 made 2 connections between statements
9 made 3 connections between statements

The 18 incorrect answers:
10 made 0 connections between statements
4 made 1 connection between statements
4 made 2 connections between statements
0 made 3 connections between statements.

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A summary of the argument analysis of puzzle I for the control group is presented in table 11.

Table 11: Argument analysis of puzzle III of control group (post-test)

No. of connections	No. of students
0	13
1	18
2	9
3	9

6.2.3 Comparison of answers of puzzle I and puzzle III

Table 12 gives a comparison of the attempts of some of the students of the experimental group. This was done in order to show their increasing ability to argue deductively.

Table 12: Comparison of answers of puzzle I and puzzle III

Puzzle I	Puzzle III
Marsha MacMahon - A statement was made that the house is next to a lake - Another statement was made thereafter that the tree in the front yard is an elm ∴ The treasure is in the kitchen.	From statement a and e, I gathered that the present is not in the attic. With the present not being in the attic the yard does not have a lawn; ∴ statement d is wiped out; ∴ I can say that the present is in the cupboard next to the stove, because the first half of statement c is not true. There is no facts to make statement b correct ∴ its redundant ∴ In my opinion the present is in the cupboard next to the stove.
Bernarain Mvondo The treasure is hidden in his boat because he says this house is next to a lake which means that the boat is ashore; The tree in the front yard represents the symbol of the boat who carries the flagpole. And kitchen, garage, flagpole are the words used by pirates.	(1) $p \Rightarrow q$ (2) $r \Rightarrow s$ (3) $t \vee u$ (4) $t \Rightarrow \sim q$ (5) p (1) & (5) q (6) (6) & (4) ~t (7) (7) & (3) u ∴ the present is in the cupboard next to the stove
Zukile Roro The treasure is in the garage. Firstly this house next to the lake so the	Statement (e) says the house is next to a main road. Using this fact we can conclude

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treasure is not in the kitchen. The treasure can not be buried under the flagpole. So it's in the garage. Statement (c) says the house is next to a lake and from statement (a) that means the treasure is not in the kitchen.	that the present is not in the attic as it is stated in statement (a) Using statement (d) I can say the yard has no lawn because from the above paragraph I know that the present is not in the attic. So this yard has no lawn. Statement (c) states that the yard has a lawn or the present is in the cupboard next to the stove. I have already proved that the yard has no lawn and now I can conclude that the present is in the cupboard. The present is hidden in the cupboard next the stove.
Phumla Thafeni The treasure is in the kitchen because this house is next to a lake and the tree in the front yard is an elm The fact that the house is next to the lake no one can guess that the treasure might be buried under the flagpole. And it is impossible that the treasure can be buried because of the lake that is next to the house. Probably inside the kitchen.	The present is in the cupboard next the stove.because statement 5 says that the house is next to the main road and that means the present cannot be found in the attic according to statement 1. Therefore the yard does not have a lawn. 1. $p \Rightarrow \sim q$ 2. $r \Rightarrow s$ 3. $w \vee t$ 4. $w \Rightarrow q$ 5. p 6. $\sim q$ 1,5 modus ponens 7. $\sim w$ 4,6 modus tollens 8. t 3,7 disjunctive syllogism 9. $p \wedge t$ 5,8 conjunctive addition
Siphokazi Ncwaiba The elm tree is in the front yard which means the kitchen is in front too. The lake is behind the house so the treasure is not at the lake. So this house next to the lake which means the kitchen is out of the picture. The tree is an elm right. In my own thinking I found out that	I found out that the present is in the cupboard next to the stove. It is stated that the house is next to a main road so it's not in the attic. Then again it says if the yard has a lawn, it is in the attic. The yard doesn't have a lawn so its not in the attic. This is a contradiction.

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statements a; b; e only state if the items are this then its there, if being a keyword so you not sure if its in the kitchen, garage or under the flagpole. While c; d state what it is. 1. kitchen – was in the lead, we know dat the tree is an elm and that its next to the lake. 2. flagpole – my conclusion comes to the kitchen or flagpole.	But I see that it's not in the study because we don't know if the house has a swing. So we left with the cupboard next to the stove. $1.\ p \Rightarrow \sim q$ $2.\ r \Rightarrow s$ $3.\ t \vee u$ $4.\ t \Rightarrow q$ $5.\ p$ $6.\ \sim q \quad 1\ and\ 5\ \text{mod}us\ ponens$ $7.\ \sim t \quad 4\ and\ 6\ \text{mod}us\ tollens$ $8.\ u \quad 3\ and\ 8\ disjunctive\,syl\log ism$
Andiswa Hazel Qosho The treasure is in the garage because it is said that if the tree in the backyard is an oak then the treasure is in the garage because a treasure can not be inside the house if the tree in the backyard is an oak	If there is a swing in the yard then the present is in the study. I think this is where the present is because if there is a swing in the yard then the present cannot be outside the house cause the swing can be the enemy and modus pones it is said that $p \Rightarrow q$ $\therefore q$ Which implies that if the swing is in the yard then the present is in the study. $1.\ p \Rightarrow q$ $2\ r \Rightarrow s$ $3.\ t \vee w$ $4.\ t \Rightarrow \sim q$ $5. \therefore p$ $1\ and\ 5$

6.2.4 Conclusions based on the analysis of puzzle I and puzzle III

The puzzles show how students learn to make connections between sentences and then deduce conclusions, which are vital tools in proving mathematical statements. 92% of students of the experimental group came to the correct conclusion with the post-test, whereas only 50% of these students came to the correct conclusion with the pre-test. Therefore there was a dramatic improvement in the number of students that could solve the puzzle. This improvement occurred after the students had received teaching on the logic of compound statements and the logic of quantified statements. Furthermore prior to receiving teaching the majority of these students could only make one or zero

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connections between statements, whereas after receiving teaching the majority of students made all three of the necessary deductions. Compare this to the control-group where the majority of students in both the pre- and post-test could only manage to make one or zero connection between and from statements. The control-group therefore showed very little improvement between pre- and post-test although they received extensive teaching on Differential Calculus in the interval between the pre- and post-test. This surely then must be an indication that knowledge of the logic of compound and quantified statements have helped to improve the ability of the experimental group of students to make logical connections between these kinds of statements.

It is also imperative that students really understand the implication of every word in a statement. Regarding this Epp **[33]** states the following: "In order to evaluate the truth or falsity of a statement, you must understand what the statement is about. In other words, you must know the meanings of all terms that occur in the statement." One of the goals of the puzzles therefore was to focus the attention of students on the implication of certain words in a statement. For example the results of the puzzles shows the growing awareness of the experimental-group of students that words like *OR* means that at least one of the two statements is true. The comparison of the two puzzles therefore show that these students are starting to read with understanding and are starting to realize that in Mathematics a word has a specific meaning. This reading with understanding is absolutely crucial in unraveling mathematical statements. Hence our teaching was geared towards enhancing, in students, this ability of reading with understanding.

6.3 Puzzles on knights and knaves

The solution to these puzzles is based on the use of proof by contradiction. In other words the student has to make an assumption and then show that the given facts contradict the assumption. This has to be done by making logical connections between facts or making deductions from facts.

This pre-test of the puzzle was given to the students without giving them any teaching on proof by contradiction. The puzzle is presented below.

6.3.1 Knights and knaves (pre-test)

The logician Raymond Smullyan describes an island containing two types of people: knights who always tell the truth and knaves who always lie. You visit the island and are approached by two natives who speak to you as follows:

A says: B is a knight
B says: A and I are of opposite type.

What are A and B?

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Solution
Suppose A is a knight.
Therefore what A says is true - (by defn. of knight)
Therefore B is a knight also – (this is what A said)
Therefore what B says is true – (by defn. of knight)
Therefore A and B are of opposite type – (this is what B said)
We have the following contradiction: A and B are both knights and A and B are of opposite type.
Therefore A is not a knight
Therefore A is a knave
Therefore what A says is false (by defn. of knave)
Therefore B is not a knight.
Therefore B is a knave also.
A and B are both knaves.

A marking memorandum consisting of the above solution was used to assess student answers.

6.3.1.1 Analysis of arguments of students

22 students attempted the puzzle.
15 students could not solve it.
7 students gave valid arguments.

Analysis of the 7 correct arguments:
5 students did the puzzle by contradiction:(Thoriso Seepi; Bernarain Mvondo; Garren Davidse; Markan Mclean; Thulani Shabangu). These students used the correct method of argument to get a contradiction. For example the following is Thulani Shabangu's argument: "*If B is indeed a knight then A is telling the truth, but then A must be a knave. But knaves always lie, therefore B cannot be a knight and A can never be a knight, since he said B is the one and B says A and I are of opposite type. Which are two statements that contradicts and knights never lie therefore they are both knaves.*"

Brian Masona used cases to solve the puzzle, but the cases are also based on contradicting an original assumption. His argument is as follows: " 1. *If A is telling the truth, that also makes him a knight. But then there would be no reason to lie on B's part if he is a knight as well. (wrong)*
2. *If B is telling the truth, it would mean he is the knight and A is the knave. However this also means A is telling truth. Therefore this is wrong.*
3. *If both are lying then A is not opposite to B and B is not a knight, so that makes them both knaves. Therefore this is true.*"
Ashwin Patience did his by process of elimination, but it is still based on contradicting an original assumption.

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Analysis of the 15 incorrect arguments:
Four students (Geraldo Maasdorp,Busisiwe Qavane, Segodi Evans, Phumla Thafeni) had some argument but their argument did not lead to the correct conclusion. For example the following is Phumla Thafeni's argument: *" B belongs to the knight people because A has stated that B is a knight. A belongs to the knaves people who always tell lies because the lie is the opposite of the truth as B stated."* Although Pumla has assumed that the first statement is correct i.e. that A is telling the truth and that therefore A is a knight, she contradicts herself by stating in the next sentence that A is a knave. This means then that she did not make the connection between her two statements which could have led to the contradiction that she needed to prove that A is a knave. The skill to induce a contradiction and then using this to draw a conclusion is therefore obviously lacking in Pumla's reasoning armour at this point in time.

Four students (Yasser Buchana, Marsha Mac Mahon, Masixole Nondumo, Siphokazi Ncwaiba) came to the correct conclusion but their supporting arguments did not support their conclusion. For example the following is Marsha Mac Mahon's argument: " *B is a knave because A always lies, therefore B is not a knight but a knave. Because B is a knave he always lies, therefore A and B are of the same type. Therefore A and B are both knaves."* Marsha does not show how she came to the first assumption "*B is a knave because A is lying"*. The rest of her argument however is logical. This shows that intuitively she knew the answer, but lacks the skills to show how she got to the answer i.e. the step-by-step reasoning that allows you to connect one fact to another or to deduce a conclusion from a fact. This step-by-step reasoning is an important skill in proving mathematical statements.
Some of the student arguments had elements of contradiction in it, but since the students did not know how to use the contradiction, they did not give a clear and concise argument to show how they reached their conclusion. For example Siphokazi Ncwaiba's argument is as follows: "*A states that B is a knight, ok if that's true it means A is a knave but the statements says knaves always lie so how do we know that A is telling the truth.*
B states that A and B are opposite types which means one is lying and the other one is telling the truth. So now who is telling the truth and who's lying. This thing revolves around the question that says who's lying and who's telling the truth.
Conclusion
A is a knave
B is a knave
It comes to this A is stating that B is a knight, if its true it automatically makes A a knave therefore a knave is a liar how sure are we that he is telling the truth."
In the very first two lines of Siphokazi's argument a contradiction arises, but since she does not recognize this, she does not use it. Since the rest of her argument depends on her using this contradiction she ends up giving a muddled argument.
A summary of the argument analysis of the pre-test of the knights and knaves puzzle is presented in table 13.

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Table 13: Argument analysis of knights and knaves pre-test

No. of connections	No. of students
0	11
1	1
2	0
3	2
4	0
5	0
6	1
7	1
8	3
9	0
10	0
11	0
12	0

The post-test was presented to students approximately 6 weeks after the pre-test.

6.3.2 Knights and knaves (post - test)

The logician Raymond Smullyan describes an island containing two types of people: knights who always tell the truth and knaves who always lie. You visit the island and are approached by two natives C and D but only C speaks.

C says: Both of us are knaves.

What are C and D?

Solution
Suppose C is a knight. Then what C says is true. - (by defn. of knight)
Therefore C and D are both knaves. – (this is what C said)
Therefore C is a knave.
Thus we have a contradiction: C is both a knight and a knave.
Hence our supposition that C is a knight is false.
Hence C is a knave.
Therefore what C says is false. – (by defn of knave)
Hence either C or D is not a knave. (De Morgan's law)
But C is a knave, hence D is not a knave.
Therefore D is a knight.

6.3.2.1 Analysis of arguments of students

29 Students attempted the puzzle.
7 gave incorrect answers.
10 gave correct answers, but had gaps in their arguments.
12 gave complete and valid arguments.

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Analysis of the 7 incorrect answers

Although the solution of these 7 students were all incorrect, they all attempted to solve the puzzle by using proof by contradiction, some with more success than others. For example Luthando Myeki argues as follows: "*The supposition is that both of them are knaves, then both C and D are lying. But I can't say what C say is a lie, because D does not say anything. Therefore there is a contradiction because C spoke and D did not. Then the supposition is false, because at least one of them must be a knight. Therefore D is a knight.*

Luthando argues that a contradiction is induced because C spoke and D did not. It is obvious therefore that he does not fully understand when and how a contradiction arises. The rest of his argument however is logically correct since he correctly negates the inferred AND with an OR. In other words he has correctly applied De Morgan's law i.e. since he initially assumed that both C *and* D are knaves, he correctly assumes the opposite that either C *or* D must be a knight after the "contradiction".

[Andiswa Qosho; Brian Masona; Luthando Myeki; Carmen Williams; Phumla Thafeni; Masixolo Nondumo; Ashwin Paulse]

Analysis of the 10 arguments with gaps

Most of these students correctly used contradiction to conclude that C is a knave, they however did not make use of this fact to advance their argument and hence to conclude that D is a knight. As a result there is a gap in their argument, since they do not show how they have reached their conclusion that D is a knight. For example Siyabonga Maki argues as follows: "*Suppose that C is a knight. Therefore C will always tell the truth. Therefore both C and D are knaves and we have a contradiction because C can not be a knight and a knave. Hence our supposition is false. Therefore C is a knave and D is a knight.*"

Analysis of the 12 correct arguments

These students presented arguments that contained all the essential steps of an acceptable solution. For example Dionisio Nunes argues as follows: "*Suppose C is a knight. Then what C says is true. Therefore C in particular is a knave. But this is a contradiction. Therefore our supposition that C is a knight is false. Therefore C is a knave. Then what C says is false. Therefore C or D is not a knave. We have already concluded that C is a Knave. Therefore D is not a knave. Hence D is a knight. Therefore C is a knave and D is a knight.*"

A summary of the argument analysis of the post-test of the knights and knaves puzzle is presented in table 14.

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Table 14: Argument analysis of post-test of knights and knaves puzzle

No. of connections	No. of students
0	4
1	0
2	1
3	0
4	1
5	3
6	4
7	3
8	5
9	3
10	3
11	2

6.3.3 Conclusions based on the analysis of knights and knaves puzzles

In both the pre and post –test the percentage of students that gave completely valid arguments are less than 50% of the total number of students that attempted the puzzles. Add to this the fact that almost 70% of students could not use contradiction to prove their argument in the pre-test then this surely is an indication that students do not find proof by contradiction an easy exercise. The fact that the majority (more than 70%) of students could induce a contradiction in the post-test however is testimony to a major improvement in proving skills as far as proof by contradiction is concerned. A comparison of some student answers to the pre- and post-test of the knights and knaves puzzles are presented in table 15.

Table 15: Comparison of answers to pre- and post-test of knights and knaves puzzles

Pre-test	Post-test
Marsha MacMahon B is a knave because A always lies. Therefore B is not a knight but a knave. Because B is a knave he always lies. Therefore A and B are of the same type. Therefore A and B are both knaves.	Suppose C is a knight. Then what he is saying is the truth. Therefore he is a knave. Which is a contradiction because C cannot be both a knight and a knave. Therefore my supposition is false. So therefore at least one of them is a knight. So suppose C is a knave. Then what he is saying is a lie. Therefore they are not both knaves. But one of them are a knave. But because C is lying, he is a knave. Therefore D is a knight. C - knave; D – knight.
Phuti Senyatsi A and B are both knights. We could say that A's statement is untrue, but we cannot	Suppose C is a knight. Therefore what C says is true. However, C says both him and

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in anyway say that B's statement is untrue because it is too ambiguous, therefore it might have some truth to it. Therefore, because there is a possibility that B's statement might have some truth, that makes him a knight. This then makes A's statement also true. A is therefore a knight as well.	D are knaves. There is therefore a contradiction because C cannot be a knight and a knave at the same time. Therefore our supposition is wrong. Suppose C is a knave. Therefore what C says is false. This means that there is at least one of them who are a knight. Therefore D is a knight and C is a knave.
Segodi Evans B said that A and him are of opposite type which means that B is right because according to the statement the two natives are completely different. A said that B is a knight. A could be wrong because on the statement Raymond said that between the two natives one is a liar. A is the one whose statement is true in either way. So it means A is the one who always tells the truth. While B's statement is not clear, he could be the one who always lies.	Suppose C is a knight. So according to our statement what C says is true. Therefore C is a knave. But C cannot be both knight and knave therefore we have a contradiction. So our supposition is false. Therefore C is a knave. So what C says is not true. Both of them are not knaves. Therefore D is a knight.
Zukile Roro B is a knight and A is a knave. A is the one who always lie. He introduces B first instead of himself. B says they are of opposite type meaning that if B is the knight then A is the knave.	Suppose C is a knight. Then what C says is true. Therefore both of them are knaves. But that is impossible, C cannot be a knight and a knave. So that is a contradiction. That means my supposition that C is a knight is false. Therefore C is a knave. If C is a knave then one of them has to be a knight. Therefore D is a knight.
Sigqibo Lande B is a knight given that A tell us that B is a knight which means both A and B are knights, but they differ in gender.	Suppose C is a knight. Therefore C tells the truth. Therefore C is telling a truth when C says that both of them are knaves. Thus there is a contradiction that you can't tell a truth and be a knave.(taking what C says). Therefore my supposition is false. Therefore C is a knave and D is a knight. (from the opposite of both). There is at least one knave and knight. I've already proven that C is a knave. That's why I say that D is a knight.

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6.4 Arguments with Quantified statements

6.4.1 Pre-test

Students had to rewrite the following statements in formal Mathematical language

Question A
1. No birds, except ostriches, are nine feet high.
2. There are no birds in this aviary that belong to anyone but me
3. No ostrich lives on mince pies.
4. I have no birds less than nine feet high.

Question B
1. All writers who understand human nature are clever.
2. No one is a true poet unless he can stir the hearts of men.
3. Shakespeare wrote Hamlet.
4. No writer who does not understand human nature can stir the hearts of men.
5. None but a true poet could have written Hamlet.

Solution
Question A
1. $\forall x$, if x is not an ostrich, then x is less than nine feet tall.
2. $\forall x$, if x is a bird in this aviary, then x is a bird which belongs to me.
3. $\forall x$, if x is an ostrich, then x does not live on mince pies.
4. $\forall x$, if x is a bird that is less than nine feet high, then x does not belong to me.

Question B
1. $\forall x$, if x is a writer who understands human nature, then x is clever.
2. $\forall x$, if x cannot stir the hearts of men then x is not a true poet.
3. Shakespeare wrote Hamlet.
4. $\forall x$, if x is a writer who does not understand human nature, then x cannot stir the hearts of men.
5. $\forall x$, if x is not a true poet, then x could not have written Hamlet.

6.4.1.1 Analysis of student answers of Quantified Statements

Question A
The majority of students did not rewrite number 1 correctly because they did not interpret the word "except" correctly and hence did not use the negation "not an ostrich" in their answer. For example Phuti Senyatsi writes: "*$\forall$ birds x, if x is nine feet high, then x is an ostrich*"
Numbers 2 and 3 was answered correctly by most students.
Number 4 was answered incorrectly by most students, since again they failed to use the universal quantifier (negation) for "no birds". For example Siphokazi Ncwaiba writes: "*$\forall$ birds x, if x belongs to me, then x is nine feet high.*"

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Question B

The majority of students rewrote number 1 correctly.

The words "no one" and "unless" caused most students to write number 2 incorrectly as they failed to use negation. For example Yasser Buchana writes: "$\forall$ *poets x, if x is a true poet, then x can stir the hearts of men."*

Most students did not realize that number 3 could not be rewritten in formal language.

The majority of students could not interpret "*no writer*" and "*does not understand*" correctly and hence rewrote it incorrectly. An example of this can be seen in Carmen Williams' answer: "$\forall$ *writers x, if x understands human nature, then x can stir the hearts of men"*

Very few students could rewrite number 5 in formal language as again they had problems with "*none*" as they did not interpret this to mean negation. Phuti Senyatsi's answer illustrates this: "$\forall$ *poets x, if x wrote Hamlet, then x is true*."

The above analysis shows that students had problems with sentences that have the negation at the beginning. Students also had problems with words like none, unless, except, etc. and could not rewrite it properly in formal mathematical language as they did not interpret the meaning of these words correctly. We therefore decided on an intervention strategy that would allow the students to correctly interpret and hence rewrite such statements correctly. The strategy included discussion and more practice exercises on the above type of exercises.

A summary of the argument analysis of the pre-test of the quantified statements are presented in table 16 and 17.

Table 16: Argument analysis of quantified statements (question 1) pre-test

Score	No. of students
0	2
1	5
2	19
3	5
4	0

Table 17: Argument analysis of quantified statements (question 2) pre-test

score	No. of students
0	1
1	12
2	15
3	3
4	0
5	0

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6.4.2 Post-test quantified statements

The following were used as a post-test:
Rewrite the following statements in formal language

1. No bank closes before 3:30 unless it is a small bank.
2. No shark eats plankton unless it is a whale shark.
3. Students never study unless they have to prepare for a test.
4. None but a true gentleman will offer his seat to a lady on a bus.
5. None but a brave soldier will fight in a war

Solution

1. $\forall x$, if x is not a small bank then x is a bank that does not close before 3:30.
2. $\forall x$, if x is not a whale shark then x is a shark that does not eat plankton.
3. $\forall x$, if x is a student that does not have to prepare for a test then x is a student who never studies.
4. $\forall x$, if x is not a true gentleman, then x is a gentleman who will not offer his seat to a lady on a bus. OR $\forall x$, if x offers his seat to a lady on a bus, then x is a true gentleman.
5. $\forall x$, if x is not a brave soldier then x is soldier who will not fight in a war. OR $\forall x$, if x fights in war, then x is a brave soldier.

6.4.2.1 Analysis of student answers of Quantified Statements

The majority of students performed quite well in the post-test. The ability to translate statements from informal to formal language and vice versa is a requisite skill necessary for forming conclusions from arguments. Furthermore some regard the universal conditional statement as the most important form of statement in mathematics. They are of the opinion that familiarity with statements of this form is essential if one is to learn to speak mathematics. It was therefore imperative for students to be well versed in this skill. A summary of the argument analysis of the post-test of the quantified statements is presented in table 18.

Table 18: Argument analysis of quantified statements (post-test)

score	No. of students
0	1
1	0
2	4
3	11
4	8
5	5

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6.4.3 Arguments with Quantified statements (forming conclusions)

The reason why these quantified statements with conclusions were done was to help students to learn how to connect statements in the correct order to form a conclusion. This is a very necessary and important skill in the proving of mathematical statements, since in proving mathematical statements one needs first of all to be able to understand where and how to start the proof. Consequently one needs to connect each statement logically with the previous one until a logical conclusion is reached.

6.4.3.1 Puzzle I (pre – test)

Reorder the premises in the following argument to make it clear that the conclusion follows logically. It may be helpful to rewrite some of the statements in if – then form and to replace some statements by their contrapositives.

1. When I work a logic example without grumbling, you may be sure it is one I understand.
2. The arguments in these examples are not arranged in regular order like the ones I am used to.
3. No easy examples make my head ache.
4. I can't understand examples if the arguments are not arranged in regular order like the ones I am used to.
5. I never grumble at an example unless it gives me a headache.

$\therefore$ These examples are not easy.

Solution

2. The arguments in these examples are not arranged in regular order like the ones I am used to.
4. If the arguments are not arranged in regular order like the ones I am used to, then I can't understand the examples.
1. If I do not understand a logic example, then I grumble at it.
5. If I grumble at an example then it gives me a headache.
3. If an example gives me a headache, then it is not an easy example.

$\therefore$ These examples are not easy.

Alternatively formal language and the rules of inference can be used:

Let:

Q = The arguments in these examples are not arranged in regular order like the ones I am used to.

P = Do not understand these examples

R = grumble at an example

S = Example give me a headache

T = It is not an easy example

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The statements in the correct order in formal language become:

2.	Q	
4.	$Q \Rightarrow P$	
1.	$\sim R \Rightarrow \sim P$	
5.	$R \Rightarrow S$	
3.	$S \Rightarrow T$	
6.	P	(from 2 & 4 – modus ponens)
7.	R	(from 1 & 6 – modus ponens)
8.	S	(from 5 & 7 – modus ponens)
9.	T	((from 3 & 8 – modus ponens)

$\therefore$ These examples are not easy

Note: Marks was allocated whenever two statements were given in the correct order, even if the other statements were not in the correct order.
Marks was subtracted if a statement was not rewritten properly in if-then form.

6.4.3.2 Analysis of student answers

A total of 20 students attempted the puzzle. Nine students (45% of the total number of students) gave a complete and valid argument. Eleven students (55% of the total number of students) presented incorrect arguments.

Fourteen students attempted to rewrite the statements in formal language and thereafter arrange them in the correct order. Seven of these students had correct answers and seven gave incorrect answers. The remaining six used the statements as they are. Two of these supplied a correct answer and four and incorrect answer.
A summary of the argument analysis of the pre-test of the quantified statements (forming conclusions) is presented in table 19.

Table 19: Argument analysis of quantified statements (forming conclusions) pre-test

No. of connections	No. of students
0	6
1	3
2	1
3	1
4	1
5	8

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6.4.3.3 Puzzle II (post – test)

Reorder the premises in the following argument to make it clear that the conclusion follows logically. It may be helpful to rewrite some of the statements in if – then form and to replace some statements by their contrapositives.

6. There is no box of mine here that I dare open.
7. My writing-desk is made of rose-wood.
8. All my boxes are painted, except what are here.
9. There is no box of mine that I dare not open, unless it is full of live scorpions.
10. All my rose-wood boxes are unpainted.

∴ My writing-desk is full of live scorpions.

Solution

2. My writing desk is made of rose-wood
5. If my writing-desk is made of rose-wood, then it is unpainted
3. If my boxes are unpainted, then it is here
1. If a box is here, then I dare not open it
4. If I dare not open a box, then it is full of live scorpions

∴ My writing-desk is full of live scorpions

Alternatively formal language and the rules of inference could be used:

Let: P = boxes that I dare open
Q = boxes full of live scorpions
R = boxes that are here
S = boxes made of rose-wood
T = boxes that are painted
U = my writing-desk

The statements in the correct order in formal language become:

2. $U \Rightarrow S$
5. $S \Rightarrow \sim T$
3. $\sim T \Rightarrow R$
1. $R \Rightarrow \sim P$
4. $\sim P \Rightarrow Q$
6. $U \Rightarrow \sim T$ (from 2 & 5 – hypothetical syllogism)
7. $U \Rightarrow R$ (from 6 & 3 – hypothetical syllogism)
8. $U \Rightarrow \sim P$ (from 7 & 1 – hypothetical syllogism)
9. $U \Rightarrow Q$ (from 8 & 4 – hypothetical syllogism)

∴ My writing-desk is full of live scorpions.

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6.4.3.4 Analysis of student answers

Twenty five students attempted the puzzle. No students used formal language and the rules of inference .Thirteen students (52% of the total number of students) gave a complete and valid argument. Five students supplied the correct order for the statements but rewrote one statement erroneously in if-then form. Three students had the correct order, but made mistakes in the rewriting of two statements. One student had the correct sequence, but made three errors in rewriting in if-then form. Three students had a completely incorrect sequence, two of which did not rewrite in if-then form and hence had a zero score.
A summary of the argument analysis of the post-test of the quantified statements (forming conclusions) is presented in table 20.

Table 20: Argument analysis of quantified statements (forming conclusions) post-test

No. of connections	No. of students
0	2
1	0
2	2
3	3
4	5
5	13

6.4.4 Conclusions based on the results of Quantified Statements:

Although the improvement from pre- to post-test was not dramatic as far as complete and valid answers are concerned, the number of students that had the correct sequence with the post-test did increase significantly (88% of the total number of students). However the results of the post-test seem to suggest that students still have some problems in rewriting statements in the correct if-then form.

6.5 Proofs

6.5.1 Set Theory

The following pre-test was administered to students:

6.5.1.1 Pre-test
For all sets A, B and C prove the following:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

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Solution

Suppose x is a particular, but arbitrarily chosen element of $A \cap (B \cup C)$,

then $x \in A$ and $x \in B \cup C$, so $x \in A$ and $x \in B$ *or* $x \in C$

hence $x \in A$ *and* $x \in B$ *or* $x \in A$ *and* $x \in C$

Case 1: $x \in A$ *and* $x \in B$

then $x \in A \cap B$, and so $x \in (A \cap B) \cup (A \cap C)$

Case 2: $x \in A$ *and* $x \in C$

then $x \in A \cap C$ and so $x \in (A \cap B) \cup (A \cap C)$

hence in either case $x \in (A \cap B) \cup (A \cap C)$

so $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

Conversely suppose $x \in (A \cap B) \cup (A \cap C)$, then

$x \in A \cap B$ or $x \in A \cap C$

Case 1 :($x \in A \cap B$)

then $x \in A$ *and* $x \in B$

since $x \in B$, $x \in B \cup C$, hence $x \in A$ *and* $x \in B \cup C$ and so $x \in A \cap (B \cup C)$

case 2: ($x \in A \cap C$)

then $x \in A$ *and* $x \in C$

since $x \in C$, $x \in B \cup C$, and so $x \in A$ *and* $x \in B \cup C$, hence $x \in A \cap (B \cup C)$

so in either case $x \in A \cap (B \cup C)$

hence $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

hence $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

NOTE: If students employed a shorter method and it was obvious that the other steps was implicit in the reasoning of the student, the student was accredited with marks for these implicit steps.

6.5.1.2 Analysis of student answers

23 students completed the pre-test. Thirteen students (57% of the total number of students) achieved a score of 30% or less and two scored 45% or less. Three of these students had a zero score, two of which supplied no answer and one (Marsha Mac Mahon) supplied a muddled answer. Marsha's attempted solution is as follows: $x \in A \cap B$ or $x \in A \cap C$

Case 1: $x \in C$

If $x \in C$ then $x \in A \cap C$ but then $x \notin A \cap B$ $\therefore x \notin C$

Case 2: $x \in B$

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If $x \in C$ then $x \in A \cap B$ but then $x \notin A \cap C$ $\therefore x \notin B$

Case 3: $x \in A$

If $x \in A$ *then* $x \in (A \cap B) \cup (A \cap C)$

$\therefore x \in A$

$\therefore A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

It is clear that the student does not know how and when to use cases. In her use of cases she attempted to show that x is not an element of sets *A, B* and *C*. This is an indication that she is not aware of what is required i.e. that she must prove $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and conversely $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

She also misinterprets intersection since she states " If $x \in C$ then $x \in A \cap C$". She is therefore definitely not cognizant of the basic definitions of set theory.

The majority of the other students in this group were aware that they were supposed to show $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and conversely $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. They however had gaps in their arguments or they used inappropriate cases leading to erroneous arguments.

Six students (26% of the total number of students) attained a score between 55% and 80%. Siyabonga's proof is as follows: Case 1: $x \in A \cap (B \cup C)$

Then $x \in A$ and $x \in B$ *or* $x \in C$ therefore $x \in (A \cap B)$ *or* $(A \cap C)$

and hence $x \in (A \cap B) \cup (A \cap C)$

$\therefore A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

Case 2: $x \in (A \cap B) \cup (A \cap C)$

then $x \in A \cap B$ or $x \in A \cap C$

then $(x \in A$ *and* $x \in B)$ *or* $(x \in A$ *and* $x \in C)$

therefore $x \in A \cap B$ or $x \in A \cap C$

therefore $x \in A \cap (B \cup C)$ and hence $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

$\therefore A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Siyabonga made inappropriate use of cases and has a circular argument in his second case. He does however use the appropriate definitions correctly and he seems to understand that he needs to show $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and conversely $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

Three of the students of this group of six applied the rules of inference in their reasoning. They showed in their margins the rule of inference that they have applied to make a deduction. (Nyirenda Pereka; Evans Segodi; Markan McLean). It is notable that these students were among the highest scorers. The following is the solution of such a student namely Evans Segodi:

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Suppose $x \in A \cap (B \cup C)$, therefore $x \in A$ and $x \in B \cup C$

$x \in A$ and $(x \in B \;\; or \; x \in C)$ but $(x \in A \quad and \; x \in B) \quad or \quad (x \in A \; and \; x \in C)$

Therefore $x \in (A \cap B) \cup (A \cap C)$

$\therefore A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

Conversely

Suppose $x \in (A \cap B) \cup (A \cap C)$ then $(x \in A \cap B)$ or $(x \in A \cap C)$

Case 1: $(x \in A \cap B)$

Then $x \in A \quad and \; x \in B$

$x \in A \quad and \; x \in B$

$x \in A$ and $(x \in B \;\; or \;\; x \in C)$ $\quad \begin{bmatrix} p \\ \therefore p \vee q \end{bmatrix}$

$x \in A \cap (B \cup C)$

$\therefore (A \cap B) \subseteq A \cap (B \cup C)$

Case 2: $x \in A \cap C$

Then $x \in A \; and \; x \in C$

$x \in A \; and \; (x \in C \; or \; x \in B)$ $\quad \begin{bmatrix} p \\ \therefore p \vee q \end{bmatrix}$

$x \in A \cap (C \cup B)$

$\therefore x \in A \cap (B \cup C)$ - by the commutative law

$\therefore (A \cup C) \subseteq A \cap (B \cup C)$

Therefore $\therefore A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Although Evans makes two errors in the final lines of case 1 and 2 it is clear that he fully understands what he is required to show. The fact that he shows in his margin that he employed disjunctive addition in order to come to the conclusion that $(x \in B \;\; or \;\; x \in C)$ must surely be an indication that the student understands that the rules of inference can be applied in such proofs. It is also perhaps indicative that the student is using the logic as a construct to reason from. If this is indeed the case, then we would have succeeded in our primary goal i.e. to get students to argue from a logic perspective.

Only two students (9% of the total number of students) managed to present a complete argument.

The results indicate that students at this point in time are not entirely comfortable with proving equality of sets by means of element arguments.
A summary of the argument analysis of the pre-test of set theory is presented in table 21.

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Table 21: Argument analysis of set theory (pre-test)

score	No of students	score	No. of students
0	3	11	1
1	1	12	1
2	0	13	0
3	1	14	1
4	2	15	2
5	3	16	1
6	3	17	0
7	1	18	0
8	0	19	0
9	1	20	2
10	0		

6.5.1.3 Set theory (post-test)

The following post-test was administered to students:

For all sets A, B and C prove the following:

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Solution

Suppose $x \in (A \cup B) \cup C$, then $x \in A \cup B \;\; or \;\; x \in C$

Case I: $x \in A \cup B$

Then $x \in A \; or \; x \in B$, then $x \in A \; or \; x \in B \; or \; x \in C$

And so $x \in A \;\; or \; (x \in B \; or \; x \in C)$ and thus $x \in A \cup (B \cup C)$

Case II: $x \in C$

Then $x \in C \quad or \quad x \in B$ and so $x \in C \; or \; x \in B \;\; or \; x \in A$

Then $x \in B \; or \; x \in C \; or \; x \in A$ and so $x \in A \; or \; x \in B \; or \; x \in C$

Then $x \in A \; or \; (x \in B \; or \; x \in C)$ and thus $x \in (A \cup B) \cup C$

So in either case $(A \cup B) \cup C \;\; \subseteq \;\; A \cup (B \cup C)$

Conversely suppose $x \in A \cup (B \cup C)$, then $x \in A \;\; or \;\; x \in B \cup C$

Case I: $x \in A$

Then $x \in A \; or \; x \in B$ and also $x \in A \; or \; x \in B \; or \; x \in C$

Thus $(x \in A \; or \; x \in B) \; or \; x \in C$

Hence $x \in A \cup (B \cup C)$

Case II: $x \in B \cup C$

Then $x \in B \;\; or \; x \in C$ and also $x \in B \; or \; x \in C \; or \; x \in A$

And so $x \in A \; or \; x \in B \; or \; x \in C$, then $(x \in A \; or \; x \in B) \; or \; x \in C$

Thus $x \in (A \cup B) \cup C$

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So in either case $A \cup (B \cup C) \subseteq (A \cup B) \cup C$

Therefore $(A \cup B) \cup C = A \cup (B \cup C)$

Note: This proof could be done using the associative law. If a student used the associative law he/she would use fewer steps. In such a case the marks would then be scaled up to 28.

6.5.1.4 Analysis of student answers

22 students completed the post-test. Seven students (32% of the total number of students) scored 32% or less. The remaining 15 students (68% of the total number of students) all scored above 53%. Twelve students (55% of the total number of students) had scores of 67% or more. Six students (27% of the total number of students) obtained a score above 80%.

Marsha MacMahon showed major improvement in her ability to prove such statements. The following is her solution:

Suppose x is a particular but arbitrarily chosen element of $(A \cup B) \cup C$, then $x \in (A \cup B) \cup C$ hence $x \in A$ *or* $x \in B$ *or* $x \in C$

Case I: $x \in A$ then

$x \in A$ then $x \in A \cup (B \cup C)$ - definition of union

Case II: $x \in B$ then

$x \in B$ then $x \in A \cup (B \cup C)$ - definition of union

Case III: $x \in C$

$x \in C$ then $x \in A \cup (B \cup C)$ - definition of union

Since in all 3 cases $x \in A \cup (B \cup C)$ we have $(A \cup B) \cup C \subseteq A \cup (B \cup C)$

If $x \in A \cup (B \cup C)$ then $x \in A$ *or* $x \in B$ *or* $x \in C$

Case I: $x \in A$ then

$x \in A$ then $x \in (A \cup B) \cup C$

Case II: $x \in B$

Then $x \in B$ then $x \in (A \cup B) \cup C$

Case III: $x \in C$

Then $x \in C$ then $x \in (A \cup B) \cup C$

Therefore in all 3 cases $x \in (A \cup B) \cup C$

Thus $(A \cup B) \cup C \subseteq A \cup (B \cup C)$

$(A \cup B) \cup C = A \cup (B \cup C)$

Although Marsha's solution has a number of gaps in it, it seems that she is now aware of what she is supposed to show i.e. $(A \cup B) \cup C \subseteq A \cup (B \cup C)$ and conversely $A \cup (B \cup C) \subseteq (A \cup B) \cup C$. She also now makes correct use of cases, whereas previously she gave the impression that she was confused as to the use of cases.

A summary of the argument analysis of the pre-test of set theory is presented in table 22.

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Table 22: Argument analysis of set theory (post-test)

No. of connections	No of students	No of connections	No. of students
0	0	15	1
1	0	16	1
2	0	17	1
3	0	18	0
4	1	19	2
5	0	20	2
6	3	21	2
7	0	22	0
8	2	23	1
9	1	24	3
10	0	25	1
11	0	26	0
12	0	27	0
13	0	28	1
14	0		

6.5.1.5 Conclusions based on the results of set theory

With the pre-test 65% of students attained a score of less than 45%, whereas with the post-test only 32% of students had a score less than 45%. The majority of students therefore obtained a score of more than 53% in the post-test. This is indicative of a major improvement in proving skills as far as proving equality of sets by means of element argument is concerned. The three students that have used the rules of inference in their arguments in the pre-test all scored above 85% in the post-test. It would appear that the students that used the rules of inference performed better than the majority of students in both the pre- and post-test, which is perhaps an indication that a thorough understanding of mathematical logic and its applications can lead to improved deducing abilities.

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6.5.2 Method of direct proof and divisibility

6.5.2.1 Pre-test

Prove the following by using the method of direct proof:
For all integers a, b and c , if a/b and a/c then $a/(b + c)$

Solution

Suppose a, b and c are particular, but arbitrarily chosen integers such that a/b and a/c
Then by definition of divide $b = ak$ and $c = al$ for some integers k and l
Then $b + c = ak + al = a(k + l)$
Let $t = k + l$, then t is an integer, since the sum of integers is an integer
Thus $b + c = at$ and hence $a/(b + c)$ - by definition of divide

The solution therefore required the students to make the following connections:

1. Use definition of divisibility
2. closure of integers under addition
3. factorize and come to conclusion

6.5.2.2 Analysis of student solutions

22 students attempted the test. Nine students (41% of the total number of students) could not make any connection and therefore had a zero score. Three students (14% of the total number of students) could only make one connection. Ten students (45% of the total number of students) made all three necessary connections.

Marsha MacMahon is an example of a student that had a zero score. Her solution is as follows:
$\forall$ integers a, b and c, if a/b and a/c then $a/(b+c)$
Suppose a, b and c are particular but arbitrarily chosen integers such that
If a/b and a/c then $a/(b+c)$

$$\frac{a}{b} + \frac{a}{c} = \frac{2a}{b+c}$$

$$\therefore \frac{2a}{b+c} \neq \frac{a}{b+c}$$

$\therefore$ *the statement is false*

Marsha assumes the part that she must prove is true. The fact that she assumes what she must prove shows that she does not understand the structure of direct proof. She commits a very basic error when she adds two fractions and adds both the numerators and denominators. This is an indication of a lower level skill that has not been completely mastered. She then concludes that the statement is false based on the fact that the two fractions are not equal.

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Siyabonga Maki is an example of a student that could only make one connection. His solution is as follows:

$\forall$ integers *a, b* and *c*, if *a/b* and *a/c* then *a/(b+c)*
therefore if *a/b* then $b = ad$ and if *a/c* then $c = ae$
therefore *a/ (b+c)*, then *a/(ad+ae),* then $\dfrac{a}{a(d+e)} = \dfrac{1}{d+e}$
we know that 1/everything
$\therefore$ our statement is true.

Siyabonga started off correctly by using the definition of divide, he however then also used what he is supposed to prove to then deduce an erroneous statement. He then used the result of this incorrect statement to conclude that the given statement is true. This shows that he knew he had to use the definition, but did not know how to use this to get to the desired result. It also shows that he does not quite understand the structure of direct proof.

6.5.2.3 Post-test

Prove the following by using the method of direct proof:
For all integers *a, b* and *c*, if *a/b* and *a/c* then *a/(b – c)*

Solution
$\forall\ a, b\ and\ c \in \mathrm{Z}\ if\ \ a/b\ \ \ and\ \ a/c\ \ then\ a/b-c$
Suppose *a, b* and *c* are particular, but arbitrarily chosen integers such that *a/b* and *a/c*
Then by definition of divide $b = ak$ and $c = al$ for some integers *k* and *l*
Then $b - c = ak - al = a(k - l)$
Now $k - l$ is an integer since integers are closed under subtraction.
Hence by definition of divide *a/ b – c* which is what was required.

The solution therefore required the students to make the following connections:

1. definition of divisibility
2. closure of integers under subtraction
3. factorize and come to a conclusion.

6.5.2.4 Analysis of student solutions

22 students attempted the test. One student made 2 connections. The remaining 21 students all gave a complete and correct solution. This implies that 95% of students gave a complete and correct solution to the post-test. Compare this to the pre-test where only 45% gave a complete and correct solution. This is a dramatic improvement, and I think indicates that almost all the students have shown some mastery of this kind of problem since even the student that had one error in his solution showed that he knew what was required by such a proof. The following is his solution:

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Suppose *a,b* and *c* are particular but arbitrarily chosen integers such that *a/b* and *a/c*.
Then $b = af$ and $c = aj$ for some integers *f* and *j*
Then $b- c = af - aj$
$b - c = a(f - j)$
Now let $f - j = l$. then *l* will be an integer and thus $b - c = al$.
It is clear that this student is cognizant of the structure of such a proof as all the indicated steps are completely correct. He however neglects to make the last deduction i.e. to deduce that $a/(b - c)$.

6.5.3 Method of direct proof and number theory

6.5.3.1 Pre-test

Prove the following statement using the direct method of proof:

For all integers *m*, if $m > 1,\ then\ 0 < \frac{1}{m} < 1$

Solution:
Suppose *m* is a particular, but arbitrarily chosen integer such that $m > 1$, then

$m \times \frac{1}{m} \rangle\ 1 \times \frac{1}{m} \quad (\sin ce\ \ m > 1)$

hence $1 \rangle \frac{1}{m}$

now $\frac{1}{m} \rangle\ 0, \quad \sin ce\ \ m > 1$

and hence $0 \langle\ \frac{1}{m} \langle\ 1$

6.5.3.2 Analysis of student solutions

22 students attempted the test. Fourteen students (64% of the total number of students) returned a zero score. Five students (23% of the total number of students) had one connection between statements. Only 3 (14% of the total number of students) students returned a complete and correct solution. Yasser Buchana who had a zero score presented the following solution:

For all integers *m*, if $m > 1,\ then\ 0 < \frac{1}{m} < 1,$

$\exists$ integers m, such that $m > 1$ and $0 < \frac{1}{m} < 1$

$m > 1$ means $1 \leq m \leq\ m + 1$

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Yasser's argument is muddled and it is clear that he is not following the structure required of this proof. Furthermore he introduces $\leq$ which shows that he is not fully acquainted with the properties of real numbers.

Siphokazi Ncwaiba who made one connection gave the following solution:

Lets say $m > 1$

$\div \frac{m}{m} \rangle \frac{1}{m}$

$1 - 1 \rangle \frac{1}{m} - 1$

$0 \rangle \frac{1}{m} - 1$

Siphokazi starts off with the correct reasoning by dividing by m on both sides of the inequality to obtain the $\frac{1}{m}$ that is required. She then attempts to show that $\frac{1}{m} \rangle 0$, but reasons along an erroneous path and ends up with the opposite of what she intended. She stops her argument abruptly when she realizes that her argument will not produce the desired result. Her solution shows however that she is aware of the structure and requirements of such a proof.

6.5.3.3 Post-test

Prove the following statements using the method of direct proof:

For all real numbers x, if $0 < x < 1$ then $x^2 < x$

Solution

Suppose x is a particular but arbitrarily chosen real number such that $0 < x < 1$

multiplying by x we get: $0 < x^2 < x, \ \sin ce \ x > 0$

hence $x^2 < x$ which is what was required.

6.5.3.4 Analysis of student solutions

19 students attempted the test. Eighteen students (95% of the total number of students) gave a complete and correct solution. One student returned a solution with one error in it. The results of the pre-test indicate that initially students struggled with this kind of proof, but the results of the post-test show that after intervention the majority of students are better equipped to deal with this kind of proof.

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6.5.4 Conclusions based on the results of direct proof

This kind of proof is based on the method of generalizing from the generic particular. These proofs are of the form "if $P(x)$ then $Q(x$") where $P(x)$ is known as the hypothesis and $Q(x)$ as the conclusion. To show that "if $P(x)$ then $Q(x)$" is true, one supposes that $P(x)$ is true and then shows that $Q(x)$ must also be true. In other words in order to prove a statement of the form "$\forall\ x \in D$, *if* $P(x)$ *then* $Q(x)$" you suppose that x is a particular but arbitrarily chosen element of D that satisfies $P(x)$, and then you show that x satisfies $Q(x)$.

Since there is such a vast number of problems that can be posed where direct proof is required one cannot conclude, based on the above results, that the experimental group of students will now be able to solve all such problems. Based on the above argument one can however make a conclusion that students are now aware of the structure and requirements of such proofs since the majority of students in both post-tests presented arguments that contained the correct structure. In other words the majority of students correctly structured their proofs by supposing x is a particular but arbitrarily chosen element of the hypothesis and then attempted to show that the conclusion is also true.

6.5.5 Method of ordinary induction (number sequences)

6.5.5.1 Pre-test

Use mathematical induction to prove that:

$2 + 4 + 6 + \ldots 2n = n^2 + n,$ for all integers $n \geq 1$

Solution

Basis step: If n = 1, then

LHS: 2(1) = 2 RHS: $1^2 + 1 = 2$

$\therefore$ LHS = RHS

$\therefore$ formula is true for n = 1

Inductive hypothesis: Suppose the formula is true for some integer $k \geq 1$,

$2 + 4 + 6 + \ldots +2k = k^2 + k$

We must show: $2 + 4 + 6 + \ldots 2(k + 1) = (k + 1)^2 + (k + 1)$

Inductive step:

Now $2 + 4 + 6 + \ldots + 2k + 2(k + 1)$

$= k^2 + k + (k + 1)$

$= k^2 + k + 2k + 2$

$= k^2 + 3k + 2$

$= (k + 1)(k + 2)$

Also

$(k + 1)^2 + k + 1$

$= k^2 + 2k + 1 + k + 1$

$= k^2 + 3k + 2$

$= (k + 1)(k + 2)$

$\therefore$ LHS = RHS

This is what was to be shown.

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6.5.5.2 Analysis of student solutions

23 students attempted the test. Four students (17% of the total number of students) attained a zero score. Six students (26% of the total number of students) achieved a score of 20%. Six students (26% of the total number of students) achieved a score of 40%. Two students (9% of the total number of students) achieved a score of 60%. Five students (22% of the total number of students) achieved a score of 100%.

All the students that had a zero score could not even start the solution and therefore wrote nothing. The majority of the students that scored 20% proved the basis step but could not proceed beyond this. Carmen Williams' solution is an example of this. The following is her solution:

LHS: $2(1) = 2$

RHS: $1^2 + 1 = 2$

S_k: $2 + 4 + 6 + \ldots 2k = k^2 + k$

S_{LH}: $2 + 4 + 6 + \ldots 2(k + 1) = (k + 1)^2 + k + 1$

$= k + 1[(k + 1) + 1]$

It seems like she had a general idea of what she is supposed to show, but lacked the ability to make the necessary logical connections between her steps.

It seems like Siphokazi Ncwaiba understood what the structure of the proof entails and therefore showed all the requisite steps but could not master the basic algebra that was required to complete the proof. The following is her solution:

1. Prove that $2 + 4 + 6 + \ldots 2n = n^2 + n$ is true for $n \geq 1$

LHS: $2n = 2(1)$ RHS: $n^2 + n = 1^2 + 1$

$= 2$ $= 2$

$\therefore$ LHS = RHS

2. Suppose that it is true for P(k)

   $2 + 4 + 6 + \ldots 2k = k^2 + k$ for $k \geq 1$

3. Use $P(k)$ to prove that $P(k + 1)$ is true for $k+1 \geq 1$

$2 + 4 + 6 + \ldots 2(k+1) = (k+1)^2 + (k+1) = k^2 + 3k + 2$

$= 2(k+1) = 2k + 2$

$= (k^2 + k) + 2 = k^2 + k + 2$

6.5.5.3 Post-test ordinary induction (number sequences)

Use mathematical induction to prove that:

$1 + 5 + 9 + \ldots (4n - 3) = n(2n - 1)$ *for all integers* $n \geq 1$

Solution

Basis step: if $n = 1$, then

LHS: $[4(1) - 3] = 1$ RHS: $2[2(1) - 1] = 1$

$\therefore$ formula is true for $n = 1$

Inductive hypothesis:

Suppose the formula is true for some integer $k \geq 1$,

$1 + 5 + 9 + \ldots (4k - 3) = k(2k - 1)$

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Inductive step:
We now need to show that it is true for $n = k + 1$

LHS: $1 + 5 + 9 + \ldots (4k - 3) + [4(k + 1) - 3]$

$= \quad k(2k - 1) + 4(k + 1) - 3$

$= \quad 2k^2 - k + 4k + 4 - 3$

$= \quad 2k^2 + 3k + 1$

$= \quad (2k+1)(k+1)$

RHS: $\quad (k+1)[2(k+1) - 1]$

$= \quad (k+1)(2k+1)$

$\therefore$ LHS = RHS
Which is what was required.

6.5.5.4 Analysis of student solutions

22 students attempted the test. Five students scored 50% or less. Two scored 30%, one scored 40% and two scored 50%.Seventeen students attained a score of 60% or more. Of these one scored 60%, one scored 70%, one scored 80%, one scored 90% and 13 achieved a 100% score.

Sigqibo Lande was one of the students that had the lowest score. His solution is as follows:
For $n = 1$

$4(1) - 3 = 1 = 1[2(1) - 1] = 1$

$\therefore$ LHS = RHS
Therefore the statement is true for $n = 1$
Suppose P(k) is true for some integer $k \geq 1$

$1 + 5 + 9 + \ldots(4k - 3) = k(2k - 1)$

We must show that it is true for k+1. Now *if* $n = k + 1$

$1 + 5 + 9 + \ldots(4k - 3) + (4k - 2) = k(2k - 1) + k(2k)$

$\therefore\ k(2k - 1) + (4k - 2)$

$2k^2 - k + 4k - 2$

$2k^2 + 3k - 2$

It is clear that Sigqibo understands what he is required to show. However he made some elementary errors and therefore could not present a concise argument. The majority of the students that did not succeed in giving a complete and correct solution also made elementary algebraic errors. This is an indication that students understand what is required, but struggle with lower level skills. In other words skills students are supposed to have mastered at school level.

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6.5.5.5 Conclusions based on the results of ordinary induction

With the pre-test 61% of students attained a score of 40% or less, whereas with the post-test only 14% of students achieved a score of 40% or less. Furthermore 59% of students achieved a 100% score with the post-test whereas only 22% of students achieved 100% in the pre-test. This surely then must be an indication that students have acquired the necessary skills as far as this kind of proof is concerned, although there is a strong indication that lower level skills need serious attention.

6.5.6 Method of strong mathematical induction (recursive sequences)

6.5.6.1 Pre-test

Suppose that c_0 , c_1, c_2,... is a sequence defined as follows:

$$c_0 = 2,\ c_1 = 4,\ c_2 = 6$$

$$c_k = 5c_{k-3} \ \textit{for all integers}\ k \geq 3$$

Prove that c_n is even for all integers $n \geq 0$

Solution

Basis step:
Choose $a = 0,\ b = 1,\ c = 2$

$c_0 = 2 = 2 \times 1,$ *which is even*

$c_1 = 4 = 2 \times 2,$ *which is even*

$c_2 = 6 = 2 \times 3,$ *which is even*

hence it is true for *a, b* and *c*

Inductive hypothesis:
Let *k* be an integer such that $k \geq 3$
Suppose *P(i)* is true for all integers *i*, with $0 \leq i \leq k$.
That is c_i is even.

Inductive step:
Now since $k \geq 3$ it follows that $k - 3 \geq 0,$ *thus* $0 \leq k - 3 < k$
then c_{k-3} is even since $k - 3 < k$
hence by definition of even $c_{k-3} = 2l$ for some integer l

$$c_k = 5c_{k-3}$$

then $= 5(2l)$

$= 2(5l)$

now $5l$ is an integer since products of integers are integers, so c_k is even.
This is what was to be shown.

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6.5.6.2 Analysis of student solutions

22 students attempted the test. Eleven students returned a zero score. Five students achieved a score of 12,5%. Three students attained a score of 50%. One student had a score of 62,5% and one student achieved a score of 75%. Hence 77% of students achieved a score of 25% or less. Only five students managed to score above 50%.

The majority of the students that returned a zero score could not even prove the basis step. Some could not even start the proof. Others of this group did have some argument, but the arguments unfortunately were mostly of the muddled variety. For example Yasser Buchana advanced the following argument:

$c_3 = 5c_{3-3} = 5\,c_0 = 5(2) = 10$

$c_4 = 5\,c_{4-3} = 5c_1 = 5(4) = 20$

$c_5 = 5\,c_{5-3} = 5c_2 = 5(6) = 30$

$\therefore true\ \ for\ \ all\ \text{int}egers\ \ k \geq 3$

Choose a = 0, b = 1

Suppose $c_i = 5c_{i-3}$ is true for integers $i \geq 3$

$c_{i+1} = 5c_{(n+1)-3}$

It seems like Yasser initially tried to check if the sequence holds true for iterations from 3 onwards. He however concludes after only three iterations that the sequence is even for all integers $k \geq 3$. Subsequently he chose basis values, but did not attempt to prove that the sequence holds true for these basis values, instead immediately after this he tries to set up a hypothesis, but then made a nonsensical deduction from this hypothesis. All of this leaves one with no choice, but to conclude that Yasser does not understand how to use strong mathematical induction to do the required proof.

Very few students made significant progress in proving the statement; however some did show that they are aware of the requirements of such proofs. In other words they attempted to apply the structure of the proof. The following solution of Evans Segodi is an example of this:

Choose a = 0 and b = 2

Basis step:

$$P(0) = c_0 = 2,\ which\ is\ even$$

$$P(1) = c_1 = 4,\ which\ is\ even$$

$$P(2) = c_2 = 6,\ which\ is\ even$$

Therefore $P(0), P(1)\ \ and\ \ P(2)$ are true.

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Inductive hypothesis: Let k be any integer $k > 2$ and suppose a_i is even for some integer i with $0 \le i < k$

Inductive step
We have $c_k = 5c_{k-3}$ for all integers $k \ge 3$
$k - 3 \ge 0$
$\therefore\ 0 \le k - 3 < k$
hence c_{k-3} is even
$5c_{k-3}$ is an even integer because it is the multiple of an even integer and so c_k is even.
Therefore c_n is even for all integers $n \ge 0$.

Evans started off correctly by attempting to prove the basis step. He however does not choose all of the desired basis values and also fails to show why the initial values hold true. He proceeds to the inductive hypothesis, but erroneously decides to let $k > 2$. Consequently he proceeded to the inductive step where he correctly showed $0 \le k - 3 < k$ and concluded that c_{k-3} is even. He deduces that $5c_{k-3}$ is even but fails to prove this by means of the definition of even. Although Evans' solution can by no means be described as a complete and rigorous proof it does have the correct structure and contain most of the elements that are required by such a proof.

6.5.6.3 Post-test

Suppose $a_1, a_2, a_3, \ldots$ is a sequence defined as follows:

$a_1 = 1, \quad a_2 = 3,$

$a_k = a_{k-2} + 2a_{k-1}$ *for all integers* $k \ge 3.$

Prove that a_n is odd for all integers $n \ge 1$

Solution
Basis step:
Choose a = 1 and b = 2, then
$a_1 = 1 = 2(0) + 1,$ *which is odd by definition of odd*
$a_2 = 3 = 2(1) + 1,$ *which is odd by definition of odd*

Inductive hypothesis:
Let k be an integer such that $k \ge 3$
Suppose *P(i)* is true for all integers i with $1 \le i < k$, that is a_i is odd.

Inductive step:
Now since $k \ge 3$ we have $k \ge 2 + 1$, then $k - 2 \ge 1$ *and* $k - 1 \ge 2 \ge 1$,
Hence $1 \le k - 2, k - 1 < k$ and so a_{k-2} *and* a_{k-1} are odd, so

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$a_{k-2} = 2l + 1 \; and \; a_{k-1} = 2m + 1$ for some integers *l* and *m*, then

$a_k = a_{k-2} + 2a_{k-1}$

$= 2l + 1 + 2(2m + 1)$

$= 2l + 1 + 4m + 2$

$a_k = 2(l + 2m + 1) + 1$

Now *l + 2m + 1* is an integer, say *p* and hence $a_k = 2p + 1$

Thus a_k is odd by definition, which is what was to be shown.

6.5.6.4 Analysis of student solutions

23 students attempted the test. One student had a zero score. Thirteen students achieved a score of 42% or less, with five of these scoring 25%. Ten students achieved a score of 50% or more, with four of these achieving a score of 75% and 3 students attaining a score of 92%. There was thus a general improvement in results, although the majority of students still scored less than 50%. Yasser Buchana again had a zero score, whereas Evans Segodi improved from 62,5% to 75%. The following is Evans' solution:

Choose $a = 1$ and $b = 2$
Basis step: *P(1)* = 1 which is odd
P(2) = *3* which is odd
Therefore it is true for *P(1) and P(2)*

Inductive hypothesis: (*P(n): a_n* is odd)
Let *k* be a particular but arbitrarily chosen integer $k \geq 2$ and suppose *P(i)* is true for all integers *i* with $1 \leq i < k$, that is a_i is odd.

Inductive step:
Given $a_k = a_{k-2} + 2a_{k-1} \; for \; all \; integers \; k \geq 3$
Since $k \geq 3$ then $k - 2 \geq 1 \; and \; k - 1 \geq 2, \; hence \, 1 \leq k - 2, k - 1 < k$
Therefore a_{k-2} is odd, $a_{k-2} = 2l + 1$ for some integer *l.*
And also a_{k-1}, is odd, $a_{k-1} = 2j + 1$ for some integer *j*

$a_k = (2l + 1) + 2(2j + 1)$

$= 2l + 1 + 4j + 2$

$a_k = 2(l + 2j + 1) + 1$

So
l + 2j + 1 is an integer, therefore c_k is odd.
This is what was required to prove.

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Although Evans erroneously chooses $k \geq 2$ in his inductive hypothesis and he does not show why the initial values are odd, this solution is more comprehensive than his solution for the pre-test. It is clear therefore that Evans understands what one needs to show in these kinds of proof.

6.5.6.5 Conclusions based on the results of strong mathematical induction

Although there was a general improvement in results, the majority of students still scored below 50%. This shows that students are still struggling to come to grips with this kind of proof. However with the pre-test half of the students had a zero score whereas with the post-test only one returned a zero score. Also only two students attained a score above 50% with the pre-test whereas ten students maintained a score of 50% or higher in the post-test. So although it seems that the majority of students still did not completely master this type of proof there was significant improvement in proving ability as far as adherence to the structure of the proof is concerned. This is corroborated by the fact that most students showed all of the necessary steps in their post-test.

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CHAPTER 7

STATISTICAL ANALYSIS OF RESULTS

7.1 Introduction

In order to determine if doing a course in logic can improve the mathematical statement proving abilities of first year mathematics students we compared the pre- and post-test scores of the logic component to the pre- and post-test scores of the different types of proofs that we dealt with in our course. As we have previously indicated the pre- and post-tests of the logic component and the proof component were our main measuring instruments. Thus in order to do this comparison the various components of the Logic tests were used to construct an overall logic score that has a possible range of 0 to 100. This was done separately for the pre- and post-tests. A similar construction was done to obtain a score for the proofs where again the pre- and post-tests were dealt with separately. Table 23 shows the terminology and the meanings used in the statistical analysis:

Table 23: Terminology and meanings used in the statistical analysis

TERM	MEANING
Logic pre	Sum of all logic pre-tests expressed as %
Logic post	Sum of all logic post-tests expressed as %
Proof pre	Sum of all proof pre-tests expressed as %
Proof post	Sum of all proof post-tests expressed as %
Logic change	Sum of logic post minus sum of logic pre
Proof change	Sum of proof post minus sum of proof pre

Since our primary interest was in examining the association between logic (as measured by the tests) and ability in proofs (as measured by those tests) we compared the following components for the experimental group: logic pre versus proof pre; logic post versus proof post; logic change versus proof change. Hence the logic scores were the assigned independent variable and the proof scores the dependent variable. The obvious reason for this is because by convention the independent variable (x) is assumed to be the one that causes or explains the variation in the dependent variable (y).

Correlation is a statistical technique used to measure the relationship between two variables. To determine whether a relationship exists between the variables of logic and ability in proofs the Pearson product moment correlation coefficient and Spearman rank correlation coefficient were used. Importantly however, these coefficients also measure the degree of relationship between the two variables. The values of these correlation coefficients range from -1.00 to +1.00, with 0.0 indicating no relationship between the variables, +1.00 indicating a perfect positive relationship and -1.00 indicating a perfect negative relationship. A positive correlation coefficient indicates that those individuals who scored high on one variable also tended to score high on the other. A negative correlation indicates that when the value on one variable is high, it will be low on the other. The closer the correlation coefficient gets to +1.00 or -1.00 the stronger the correlation and the closer it gets to 0.00 the weaker it is. The Pearson coefficient is a

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parametric test whereas the Spearman coefficient is a non-parametric test. Non-parametric tests are tests where the dependent variables are ranks i.e. where the data are ranked according to some criteria, whereas for parametric tests the data need not be ranked. Spearman is a special case of the Pearson product moment and is most often used when the number of pairs of scores is less than 20, which was the case for some of our data. The Spearman test utilizes ranked scores and in our case the scores were ranked from low to high. If a positive association exists, we would expect to see a positive correlation in the logic and proof scores, both pre and post. It is also reasonable to expect that a positive change in logic scores would be positively correlated with a positive change in proof scores. In other words if a student improved in logic then that improvement would result in an improvement in ability in proving mathematical statements. Besides the fact that we need to determine if there was a relationship between the variables we also needed to check if the relationship (if there is one) occurred by chance or not. Measures of *statistical significance* tell us the *probability* that the association occurred by chance. We will check for statistical significance at the *.05 level.* Significance at the .05 level means that only 5 times out of 100 the results obtained occurred by chance alone, therefore the probability that it occurred by chance is at the 5% level. Hence any probability less than .05 will be accepted as significant.

The null hypothesis for both Pearson and Spearman is $\rho = 0$. In other words the null hypothesis is that there is no relationship between the two variables.
The low number of observations can be ascribed to the fact that not all students completed the tests. Only those students that had all the relevant data could be taken into consideration. Since the number of pairs of scores was lower than 20 the Spearman test is the preferred test.

7.2 Logic pre versus Proof pre

The Pearson correlation coefficient for **logic pre versus proof pre** is 0.35768 and for Spearman it is 0.31098 indicating in both cases a weak positive relationship. Both correlation coefficients are also *not* statistically significant at the .05 level since both have probabilities much greater than .05. Table 24 contains the result for logic pre versus proof pre:

Table 24: Comparison of Logic pre to Proof pre

Pearson correlation coefficient	0.35768
Probability of significance	0.3102
Number of observations	10
Spearman correlation coefficient	0.31098
Probability of significance	0.3818
Number of observations	10

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7.3 Logic post versus Proof post

For **logic post versus proof post** the Pearson coefficient is 0.18250 and for Spearman it is 0.21366 which indicates a very weak positive relationship. The correlation coefficients are *not* significant since it is greater than .05. The result for logic post versus proof post is shown in Table 25:

Table 25: Comparison of Logic post to Proof post.

Pearson correlation coefficient	0.18250
Probability of significance	0.5702
Number of observations	12
Spearman correlation coefficient	0.21366
Probability of significance	0.5049
Number of observations	12

7.4 Logic change versus Proof change

For **logic change versus proof change** the Pearson coefficient is 0.42407 and the Spearman is 0.35000 which also indicates a weak relationship. The correlation coefficients are *not* statistically significant at the .05 level since both have probabilities much greater than .05. The result for logic change versus proof change is shown in table 26:

Table 26: Comparison of Logic change to Proof change

Pearson correlation coefficient	0.42407
Probability of significance	0.2553
Number of observations	9
Spearman correlation coefficient	0.3500
Probability of significance	0.3558
Number of observations	9

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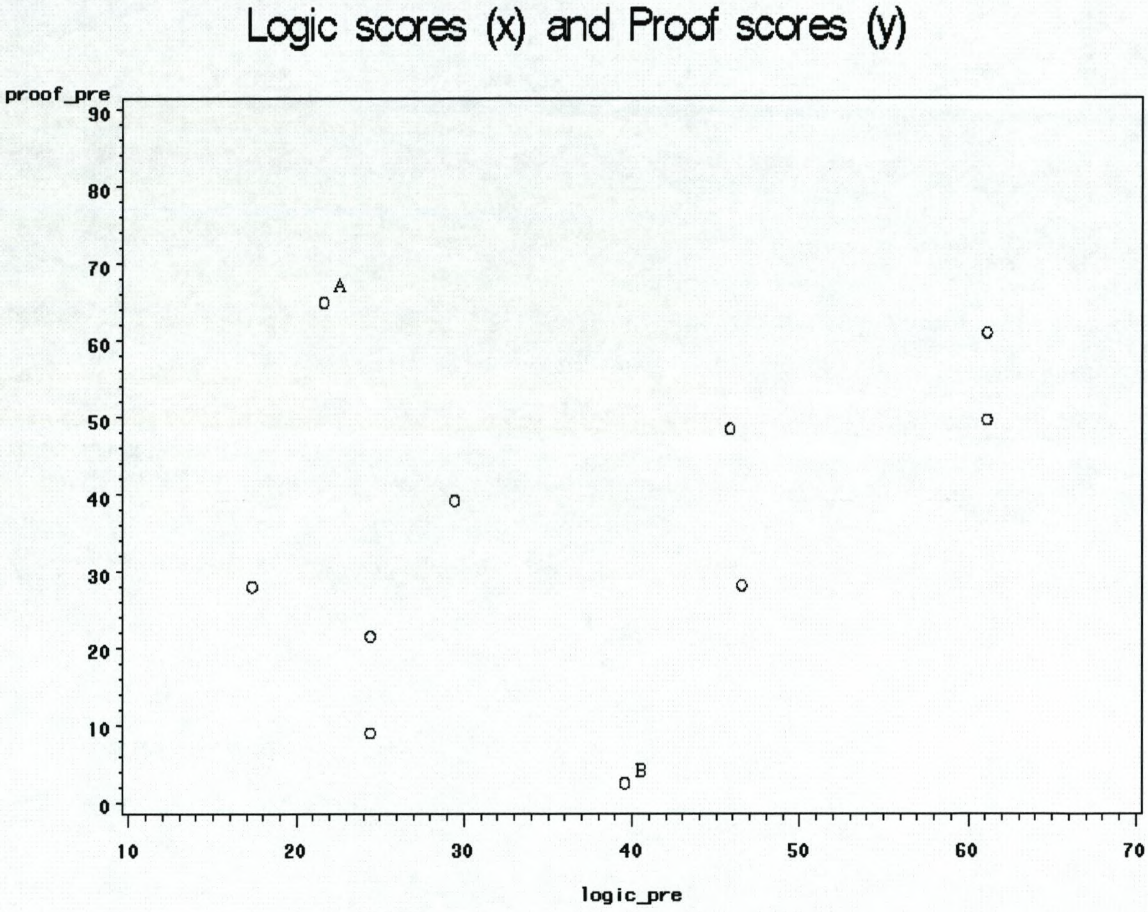
7.5 Graphs

A scatter diagram can be used to determine if an association exists between two variables. The amount of scatter in a scatter diagram gives us a rough measure of the strength of a correlation and normally the diagram is done first and then the correlation coefficient is determined to either deny or confirm the findings of the scatter diagram.

7.5.1 Logic pre versus Proof pre

The scatterplot of logic pre versus proof pre is consistent with the correlation coefficient i.e. it shows a weak positive linear relationship. However if points A and B could be ignored then the amount of scatter would be much less and hence the degree of association much more. The scatterplot for logic pre versus proof pre is shown in figure 12.

Figure 12



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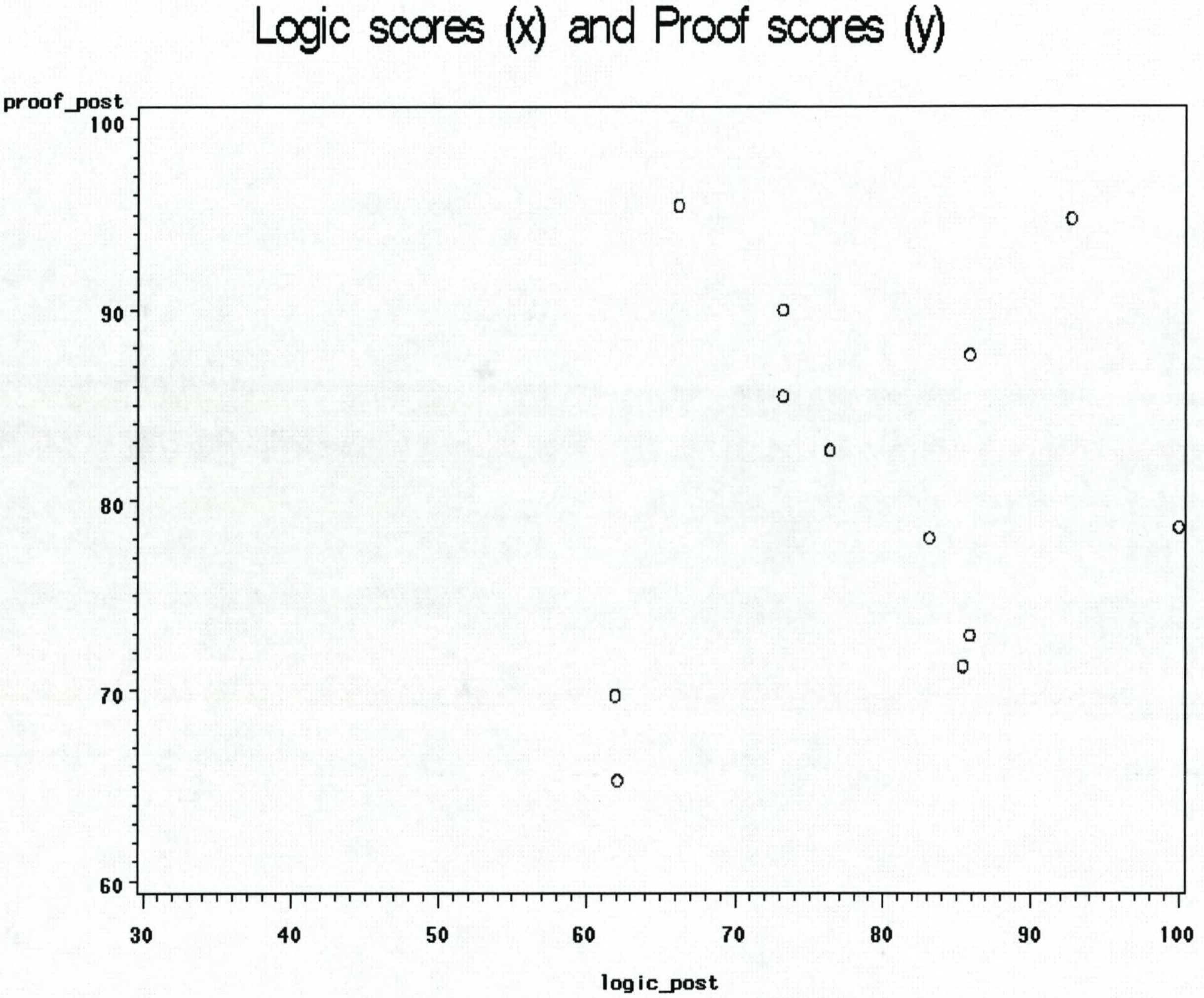
7.5.2 Logic post versus Proof post

The scatter diagram of logic post versus proof post seems to indicate a non-linear relationship. However all scores for both logic and proof are above 60% whereas the majority of scores for logic pre and proof pre were below 50%. The scatter plot for logic post versus proof post is shown in Figure 13.

Figure 13



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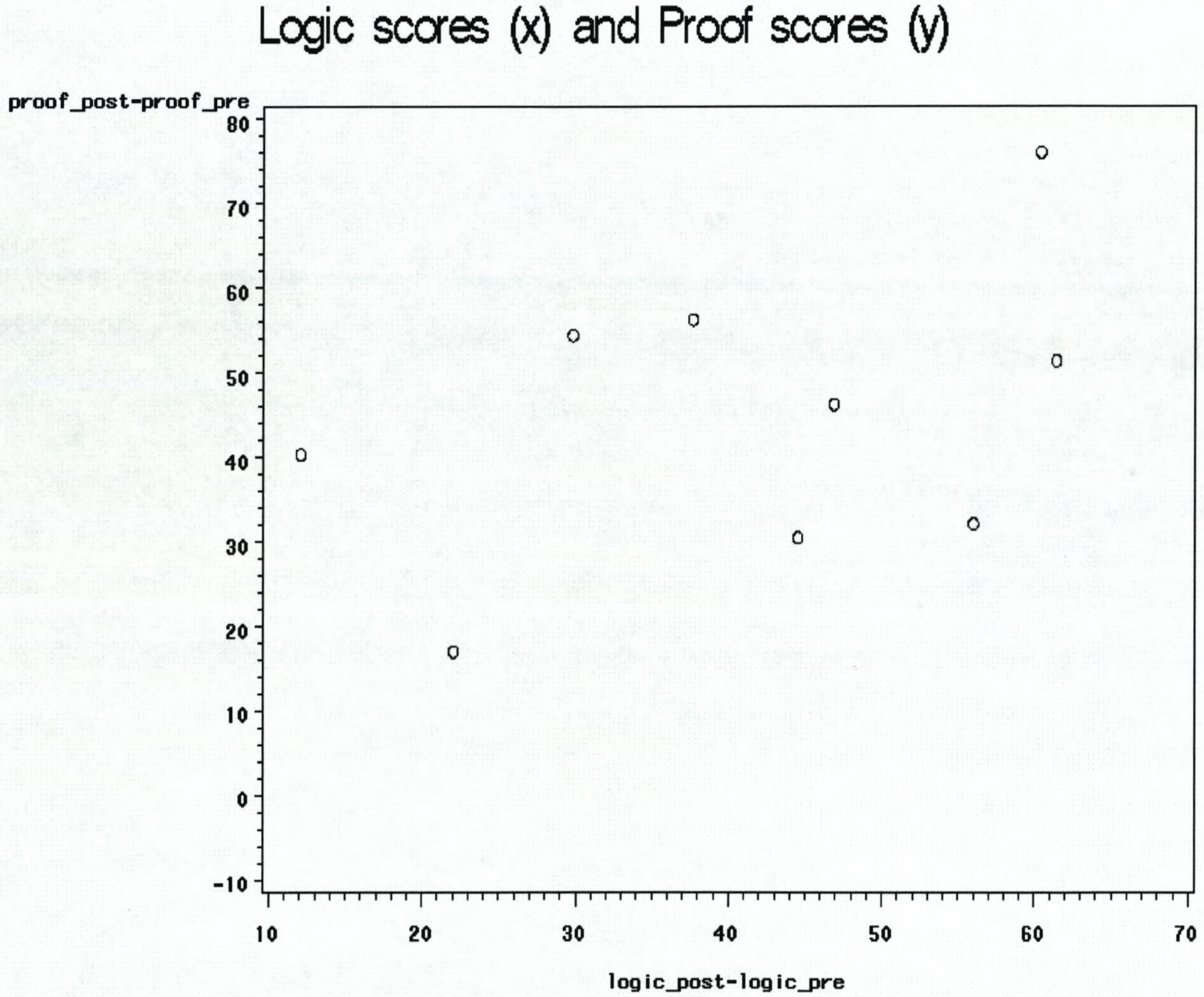
7.5.3 Logic change versus Proof change

The scatter diagram of logic change versus proof change indicates a weak positive linear relationship which confirms the correlation coefficient. The distribution of the scatter plot indicates that there is an even split between those with a positive correlation between logic difference and proof difference and those with a negative correlation. The scatterplot therefore indicates that there is no conclusive evidence for a positive association since it exhibits a weak positive linear relationship. The scatterplot for logic change vs proof change is shown in Figure 14.

Figure 14



The fact that two variables are correlated does not imply causality; conversely if two variables are not correlated one cannot be the cause of the other. Since in all three cases above a weak relationship was indicated there is thus no clear indication that logic had an effect on proving ability. So, despite the fact that there *is* indication of a weak positive relationship in at least two of the coefficients the null hypothesis has to be accepted.

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7.6 Control Group versus Experimental Group

We also compared the scores of the experimental and control groups in terms of the first logic puzzle (pre-test) and the third logic puzzle (post-test). Our first comparison was at the baseline (pre-test) where the expectation was that the two groups would be similar. On this component the scores ranged from 0 to 3. The results of this comparison are shown in Table 27 below:

Table 27: Comparison of Pre-test scores of Control and Experimental groups

PUZZLE I (pre-test)					
Control group					
Outcome variable	**0**	**1**	**2**	**3**	**Total**
Frequency	12	5	3	1	21
Percent	31.58	13.16	7.89	2.63	55.26
Row percent	57.14	23.81	14.29	4.76	100
Experimental group					
Frequency	7	7	1	2	17
Percent	18.42	18.42	2.63	5.26	44.74
Row percent	41.18	41.18	5.88	11.76	100
Total	19	12	4	3	38
Total percent	50.00	31.58	10.53	7.89	100.00

Frequency missing = 5

The statistical results based on Table 27 are shown in table 28 below:

Table 28: Statistical results based on table 27

Statistics for Puzzle I			
Statistic	**Degrees of freedom**	**Value**	**Probability**
Chi-square	3	2.5901	0.4592
Likelihood ratio chi-square	3	2.6308	0.4521
Mantel-Haenszel chi-square	1	0.4920	0.4830
Phi coefficient		0.2611	
Contingent		0.2526	
Cramer's V		0.2611	
Fisher's Exact test:	Table probability Probability		0.0166 0.5109

The following is an explanation of the values in Table 27. The *outcome variable* is the score obtained by the students. The *frequency* indicates the number of students that obtained a score. For example 12 of the control group and 7 the experimental group of students scored 0 for the first puzzle. The 21 at the end of the frequency row is the total number of control group students that participated in the study and the 17 in the frequency row of the experimental group is the total number of students for the experimental group. The 38 indicates the total number of students in both the control and experimental groups.

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The *percent* row indicates the percentage of students that attained a specific score. For example the 31.58% of the control group shows that 12 out of 38 (expressed as a percent) students scored a 0 for puzzle 1.
The *row percent* indicates the number of students of that specific group that obtained a specific score. For example 7 out of 17 students (41.18%) of the experimental group scored a zero on puzzle 1.

Since more than 50% of the cells have expected counts less than 5 the chi-square statistic might not be valid and therefore Fisher's exact test is preferred. If the probability of the Fisher's exact test is less than 0.05 then there will be a significant difference between the puzzle scores of the control and experimental groups. Since p = 0.5109 for the Fisher test, the groups do *not* differ significantly. Therefore at the beginning of the study the abilities (to solve logic puzzles) of the control and experimental groups of students were similar.

As already indicated our second comparison is on the post-test scores to see if the experimental group performed better as a result of having been taught the content in the logic course. The results of this comparison are shown in Table 29:

Table 29: Comparison of Post-test scores of Control and Experimental groups

PUZZLE III (post-test)					
Control group					
Outcome variable	**0**	**1**	**2**	**3**	**Total**
Frequency	5	7	7	2	21
Percent	11.90	16.67	16.67	4.76	50.00
Row percent	23.81	33.33	33.33	9.52	100
Experimental group					
Frequency	1	1	4	15	21
Percent	2.38	2.38	9.52	35.71	50.00
Row percent	4.76	4.76	19.05	71.43	100
Total	6	8	11	17	42
Total percent	14.29	19.05	26.19	40.48	100.00

Frequency missing = 1
The statistical results based on Table 29 are shown in Table 30 below:

Table 30: Statistical results based on Table 29

Statistics for Puzzle I			
Statistic	**Degrees of freedom**	**Value**	**Probability**
Chi-square	3	17.9260	0.0005
Likelihood ratio chi-square	3	20.0535	0.0002
Mantel-Haenszel chi-square	1	14.5871	0.0001
Phi coefficient		06533	
Contingent		0.5469	
Cramer's V		0.6533	
Fisher's Exact test :	Table probability Probability		4.002×10^{-6} 1.524×10^{-4}

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The row percent indicates that the percentage of control group students that achieved a perfect score have increased from 4.76% to 9.52%, whereas 71.43% of the experimental group scored full marks for the puzzle after the intervention.
Since 50% of the cells have expected counts less than 5 the chi-square statistic may not be a valid test, hence Fisher's exact test is preferred. The $p = 1.524 \times 10^{-4}$ for Fisher's test which is substantially less than 0.05, hence the experimental and control groups differ highly significantly.

7.7 Stratified Analysis

Using Cochran-Mantel-Haenszel methodology a more meaningful second comparison was done on the post-test scores. This second comparison utilized a stratified analysis using the pre-test scores as strata. What this in essence means is that the pre-test scores of both the control and experimental group of students are compared to their post-test scores. For example all the students that scored 0 in the pre-test are checked to see what they scored in the post-test i.e. did they score the same, less or was there an improvement. The following four tables will investigate this.

7.7.1 Controlling for a pre-score of 0

Table 31 shown below reports on students that scored 0 in the pre-test. This 0 score is then compared with the post-test i.e. how did these students perform in the post-test. The following is an explanation of the values in table 8. The *outcome variable* is the new score obtained by the students. Four of the 6 students (i.e. 66.67% of students) of the experimental group that scored 0 in the pre-test scored full marks in the post-test. Only one of the six (16.67%) again scored 0. The same trend is not seen in the control group where zero students scored full marks and 41.67% of students again scored 0 and 50% had improved to a score of 1.

Table 31: Controlling for a pre-score of 0

Control group					
Outcome variable (score)	**0**	**1**	**2**	**3**	**Total**
Frequency	5	6	1	0	12
Row percent	41.67	50.00	8.33	0.00	
Experimental group					
Frequency	1	0	1	4	6
Row percent	16.67	0.00	16.67	66.67	
Total	6	6	2	4	18

7.7.2 Controlling for a pre-score of 1

Table 32 shown below reports on students that scored 1 in the pre-test. Three of the five control group students (i.e. 60% of the students) that scored 1 in the pre-test scored 2 in the post-test and 40% of these students now scored full marks. The experimental group shows a dramatic improvement as 71.43% of these students scored full marks in the post-test.

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Table 32: Controlling for a pre-score of 1

Control group					
Outcome variable	**0**	**1**	**2**	**3**	**Total**
Frequency	0	0	3	2	5
Row percent	0.00	0.00	60.00	40.00	
Experimental group					
Frequency	0	0	2	5	7
Row percent	0.00	0.00	28.57	71.43	
Total	0	0	5	7	12

7.7.3 Controlling for a pre-score of 2

Table 33 shown reports on students that scored 2 in the pre-test. The control group of students did not show an improvement, in fact one of these students scored one in the post-test which is worse than the two scored in the pre-test. The remainder of these students scored a two again and hence did not improve. In contrast the experimental group student that scored two improved to three in the post-test.

Table 33: Controlling for a pre-score of 2

Control group					
Outcome variable	**0**	**1**	**2**	**3**	**Total**
Frequency	0	1	2	0	3
Row percent	0.00	33.33	66.67	0.00	
Experimental group					
Frequency	0	0	0	1	1
Row percent	0.00	0.00	0.00	100.00	
Total	0	1	2	1	4

7.7.4 Controlling for a pre-score of 3

Table 34 shown below reports on students that scored 3 in the pre-test. The control group student that had a perfect score in the pre-test regressed to 2 in the post-test. The two experimental group students maintained their perfect score.

Table 34: Controlling for a pre-score of 3

Control group					
Outcome variable	**0**	**1**	**2**	**3**	**Total**
Frequency	0	0	1	0	1
Row percent	0.00	0.00	100.00	0.00	
Experimental group					
Frequency	0	0	0	2	2
Row percent	0.00	0.00	0.00	100.00	
Total	0	0	1	2	3

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7.7.5 Cochran-Mantel-Haenszel statistics

Table 35 is a summary of the statistics for the stratified analysis. Table 35 reports on the Cochran-Mantel-Haenszel test for Puzzle III controlling for Puzzle I. The Row Mean Scores differ significantly at a 1% level of significance ($p = 0.0013$ therefore $p < 0.01$). The significant p-value indicates that the scores obtained by the experimental and control groups are conditionally dependent. It could be seen that the experimental group improved their prior scores whereas this was not the case for the control group.

Table 35: Cochran-Mantel-Haenszel Statistics (Modified Ridit Scores)

Statistic	Alternative Hypothesis	DF	Value	Probability (p)
1	Nonzero Correlation	1	10.0135	0.0016
2	Row Mean Scores Differ	1	10.3371	0.0013
3	General Association	3	15.6618	0.0013

Frequency missing = 6

7.8 Conclusions based on the comparison between control and experimental groups

The statistical results for the comparison show that the initial deductive abilities of control and experimental groups were more or less similar. Subsequently the experimental group was exposed to instruction in a course on logic, whereas the control group received instruction in a course on Differential Calculus. Approximately 5% of control group students and 12% of the experimental group attained a perfect score in the pre-test. With the post-test these figures have increased to approximately 10% and 71% for the control and experimental groups, respectively. So the number of control group students that achieved a perfect score had doubled whereas the number of experimental students had increased six-fold. The stratified analysis also showed that the experimental group improved consistently, whereas the control groups were inconsistent and did not show the same dramatic improvement as the experimental group. It is obvious that the remarkable improvement of the experimental group as far as the logic puzzles are concerned can only be ascribed to the instruction in logic.

7.9 Summary of conclusions on statistical analysis

The statistical analysis reveals that there is no clear indication that knowledge of logic ameliorates mathematical statement proving ability. The above discussion however makes it clear that the experimental group performed better than the control group in the logic post-test as a consequence of instruction in logic. Since the study did not compare control and experimental groups with respect to proving mathematical statements, we do not know if knowledge of logic would have caused the experimental group to perform better than the control group in this aspect. As a result we can only conclude that knowledge of logic does contribute to students making more deductions between and from statements of logic puzzles, than students that have no knowledge of logic.

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CHAPTER 8

SUGGESTIONS, RECOMMENDATIONS, CONCLUSION and FUTURE RESEARCH

8.1 Introduction

The International Commission on Mathematical Instruction (ICMI) **[48]** in their discussion document on the role of proof and proving in mathematics education states the following: "The recent NCTM[35] Principles and Standards document has elevated the status of proof in school mathematics, as have several European educational jurisdictions responsible for the school mathematics curriculum (in the UK, Italy, Spain and elsewhere). Some developing countries like South Africa also specifically mention the teaching of other functions of proof such as explanation and the importance of making and proving or disproving conjectures." Stylianides and Stylianides **[80]** shares these sentiments as can be seen from the following quote: "There are currently increased efforts to make proof central to school mathematics throughout the grades." I think it is clear from the above quotes and the earlier discussion on the importance of proof (pg 21) that proof and proving is essential in mathematics. However despite the importance of proof most students of mathematics are not adequately prepared by the school system for proof and proving at tertiary institutions.

8.2 Phases of Proof

A number of researchers are of the opinion that students are ill-prepared for the rigours of proving at tertiary level by the proving methods presented to them at school. One of the reasons advanced for this state of affairs is that students are only presented with the phases of exactification and application of proof. In other words it has become customary to teach proof in mathematics by deductively presenting the proof and then asking students to learn it and then use it to solve homework and examination problems. Some like Kleiner **[52]** believe that this deductive presentation might lead students to develop feelings of inferiority since this might lead students to believe that proofs are created by geniuses who start with axioms and reason directly and flawlessly to theorems. In other words students do not associate struggling and failure with proving. Kort et al **[55]** therefore argue that teachers fail to teach students that feelings associated with various levels of failure are normal parts of learning and that these feelings can actually be helpful signals for how to learn better.

The other phase (the phase of experimentation) involved in proving was not previously shown to students at school level in South Africa. The National Curriculum Statement (NCS)[36] makes an attempt to change this state of affairs by proposing an investigative approach to proving. In this approach the phase of experimentation is represented by the investigation. From the results of the investigation learners are expected to make

[35] NCTM – National Council of Teachers of Mathematics.

[36] The new South African mathematics curriculum for grades 10 to 12.

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conjectures and generalisations. Subsequently learners are expected to provide explanations and justifications and attempt to prove conjectures.

Students only start developing deductive reasoning skills on the third of the five Van Hiele levels of reasoning. This I believe is an indication of the difficulty level of deductive reasoning.The Van Hieles attributed the dismal performance by students in geometry to the fact that the curriculum was presented at a higher level than those of the students. As a result the students could not understand the teacher and the teacher could not understand why the students could not understand. Is this perhaps also the case with other forms of proof i.e. the fact that proofs are presented to students initially on too high an abstract level and hence they cannot understand? We are of the opinion that this is the case hence what is needed is to start proofs from the phase of experimentation so that students can start the proof with their pragmatic reasoning schemas as proposed by Cheng et al **[14]** and agreed to by Stylianides and Stylianides **[80]**. If this methodology is followed students will not be required to go straight to deductive reasoning, but will start with inductive and other forms of reasoning where they can use their pragmatic reasoning schemas that form part of their natural reasoning abilities. This therefore will allow students to start with less abstract and deductive skills which are less demanding and hence allow students to be phased into proving initially using skills they are more competent in. This methodology might not always be practically possible because of time constraints. We however strongly advocate that this is the methodology that should be followed when students are introduced to proof and proving for the first time, since in this way students will not only understand the need for the proof but would have seen the development of the proof through all the different phases and in this way perhaps perceiving that the need for proof is based in practical considerations. The question is does the investigations prescribed by the NCS mentioned earlier fulfill this role?

My experience has been that the majority of mathematics teachers and advisors in South Africa are not aware of the different phases of proving and the need for it. They therefore do not understand the need for investigations and adhere to it only since it is a prescribed assessment method. The result is that the deductive presentation of proofs is the preferred pedagogical method and the investigation is done in a very perfunctory manner. The consequence of this deductive presentation is that the majority of proofs are presented to learners on an abstract level that is beyond most of them since none of their studies prior to this has adequately prepared them for the rigours of deductive reasoning. Also our research has shown that the majority of teachers are not aware of the different types of proof and do not even know what types of proof are part of the school curriculum (i.e. the proofs they are teaching). This state of affairs should be addressed as a matter of urgency especially seen in the light of the renewed emphasis on proof and proving. A possible solution for this problem is to expose teachers to instruction in proof that include all the different types of proof, structure of proofs, phases of proof, deductive reasoning, elementary logic and the development of abstract reasoning in learners. I am of the view that even the contested views as to what proof is should be included in such instruction. This instruction should be for both prospective teachers and most importantly also for teachers already part of the educational system.

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8.3 Logic and Deductive reasoning

Although our research has not conclusively shown that knowledge of logic improve the proving abilities of students it is clear that logic does form the basis of the majority of the proofs that we dealt with. One way of describing deductive reasoning according to Stylianides and Stylianides **[80]** is as the general form of reasoning associated with logically necessary inferences based on given sets of premises. They regard the logical rules of inference modus ponens and modus tollens as two specific forms of deductive reasoning. According to them modus ponens forms the foundation of direct proof and the proof method by mathematical induction, whereas modus tollens is the foundation of indirect proof which includes proof by contradiction and contraposition. Epp **[34]** is of similar opinion as can be seen from the following quote: "It is clear that proof and deductive reasoning are very closely associated. Deductive reasoning in turn cannot be separated from logic." The above and some of our earlier arguments show that logic, deductive reasoning and proof are intrinsically linked. It is imperative therefore that teaching and learning of proof should include instruction in logic.

We are of the opinion that learners at high school level should at least be introduced to the rules of inference since they utilize the rules of inference in the majority of their proving exercises. For example the type of proof that is prevalent in the high school system in South Africa is direct proof. In this proof method one argues from a hypothesis and make deductions until you reach a conclusion. This proof method therefore relies heavily upon deductive reasoning which in turn is mostly based on the rules of inference. Furthermore the following quote from Stylianides et al **[80]** shows how strong the relationship between deductive reasoning and proof is: "In addition, from a psychological standpoint the development of students' ability for deductive reasoning has been found to go along with the development of their ability for proof." These I firmly believe are the most compelling reasons why the rules of inference should be introduced as a topic of study at school level. We believe that this can be done less formally by utilizing logic puzzles. The logic puzzles presented to students in our study contained the following rules of inference: modus ponens, modus tollens and disjunctive syllogism. Hence puzzles like these can be utilized as an informal introduction to deductive reasoning to show students how in proving to make deductions using these rules of inference.

Since reasoning by deduction plays such a crucial role in proving I think it is important that much more research is done in this area utilizing the findings of neuroscience and cognitive psychology so as to gain a deeper and better understanding of this type of reasoning. Studies done by Cheng et al **[14]** has shown that deductive reasoning does not happen naturally so we need to research and find teaching methodologies that will lead to the enhancement and development of this kind of reasoning by students. The results of our study have shown how students have improved in their ability to make connections (deductions) between statements and to draw conclusions from given hypothesis in logic puzzles as a result of exposure to the rules of inference. The skills of making connections between statements, drawing conclusions and reasoning one-step at a time is vital skills in proof and proving. We therefore emphasize again that Lewis-Carol type logic puzzles can be utilized as an informal way of introducing students to deductive reasoning and as a

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tool to make students aware of the implications of logical connectives such as *and*, *or* and *not* in mathematical statements. We must therefore pay much more attention to the reasoning, psychological and neural processes that are involved in deductive reasoning to improve proof and proving at all levels of education.

8.4 Logic and cues

Cheng et al **[14]** are convinced that instruction in logic should be accompanied by concrete examples to show where and when the logic can be applied in proofs as can be seen from the following quote: " …if logic instructors wish to influence their students' inferential behaviour in the face of novel problems, they must do much more than they currently do to show how to apply logical rules to concrete problems." This therefore mean that if students are instructed in logic without showing them how to use the logic in concrete examples of proving then the training will not be beneficial. This is perhaps why our hypothesis was not confirmed since we did not consistently show where the logic can be applied in specific proving situations. We therefore suggest that the teaching of logic principles be accompanied by examples in proof where the logic is applied. The opposite should also be done i.e. whenever proof is taught then cues should be given as to where and when it is possible to apply the logic principles.

8.5 A Hierarchical and Sequential System for proof

Proof should be done right through the grades as one of the concerns of a number of researchers has been that students are abruptly introduced to proof in high school and that this is the reason why students struggle so much with proof. Proof and proving therefore should form an integral part of all mathematics starting from the lower grades and continuing through all levels of school and tertiary institutions. It is important that instructors in proof should be cognizant of the different levels of reasoning involved in proof. Proof curricula should therefore be devised in such a way so as to allow students to go through the different levels of reasoning in a hierarchical and sequential manner. Care should be taken that the level of abstractness of a proof matches the level at which the student can reason abstractly. Proof instructors should therefore based on these reasoning levels devise proof curricula that emphasize a hierarchical and sequential approach that takes into consideration student abstract reasoning development.

The South African revised national curriculum statement (RNCS) for grades R to 9 includes some of the building blocks of the reasoning and skills involved in proving such as: conjecturing, inferring, deducing, justifying refuting, forming conclusions, etc. This shows that the curriculum writers attempted to design the curriculum in such a way as to allow the learners to engage with proof or aspects of proof in a coherent way right through the grades. Our contention however is that these proof skills are not learned since learners are struggling to master the basic numeracy skills (as indicated earlier) without which they cannot proceed to more abstract reasoning skills and as a result the efficacy of the design is negated.

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8.6 Proof structure and practice

Stylianides et al **[80]** argue that practice plays an important role in students' ability to do deductive reasoning and thus in their ability for proof. They state the following in this regard: "A possible use of practice is to help students develop the strategies for effective management of their working memory capacity. In addition, we hypothesize that practice can be used to help students internalize the general logical structure of different proof methods, such as proof by contradiction, thus releasing working memory capacity to be spent in the application of these proof methods." Since I fully concur with this argument of Stylianides et al I will use an example of proof by contradiction as an explication.

The logical structure of proof by contradiction is as follows:

Step 1: Suppose the statement to be proved is false (i.e. use negation)
Step 2: Show that this supposition leads logically to a contradiction
Step 3: Conclude that the statement to be proved is true.

Example
Prove the following statement by contradiction:
For all integers a, b and c, if $a \nmid bc$ then $a \nmid b$

Solution

Step 1
Suppose $\exists$ integers a, b and c such that $a \nmid bc$ and a/b

Step 2
Since a/b there exists an integer k such that $b = ak$ - by definition of divide
Then $bc = (ak).c$
$bc = a(k.c)$ - by associative law
But kc is an integer since it is a product of integers and so a/bc - by definition of divide
Thus $a \nmid bc$ and a/bc which is a contradiction

Step 3
Hence the supposition is false and the statement is true.

8.6.1 Discussion

What Stylianides et al **[80]** is suggesting is that the student needs to make the logical structure of proof by contradiction (as set out above) part of his/ her long term memory. In order to do this example the student also needs to have internalized the definition of divide and the associative law. The working memory can then access this information from the long term memory to make the indicated connections. The following delineates how this happens in practice:

- First step one has to be utilized to write the negation. This implies that the student at an earlier stage has engaged with negations and now has a complete mental schema

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of negation. This means that the student knows that negation in this case means accepting that the hypothesis is true i.e. $a \nmid bc$ and the conclusion is false i.e. *a/b*

- Second the student should know from prior experience that he/ she next has to start with *a/b* since the definition of divides is known. This definition should then be applied.
- Next the equation that is a result of the application of the definition should be connected to the hypothesis i.e. by multiplying by c on both sides of the equation giving $bc = (ak).c$ Again the student can only know this as a result of prior experience that is brought about by practice.
- The student should then realize that the definition of divide can be applied again to the equation $bc = (ak).c$ but this time in the opposite direction. In other words if the associative law is applied to give $bc = a.(kc)$, then from this equation one can deduce that *a/bc.* This deduction can only be done by comparing the equation to the definition of divide that is stored in the long term memory.
- The fact that $a \nmid bc$ and *a/bc* brings about the contradiction that was required.
- Step 3 can now be applied to conclude that the supposition is false and hence the original statement is true.

It is clear from the above outline that each step in the reasoning process is dependent on the previous step.

Now if the student has not internalized the structure of the proof then he/ she will have to keep in the working memory the required definitions and the structure of the proof while at the same time devising a strategy to induce a contradiction. This obviously limits working memory capacity and is a much more difficult exercise and we therefore suggest that the student internalize the structure of the proof by means of practicing on examples that require few deductions. As the student becomes more confident in applying the structure of the proof progressively more difficult examples can be provided. This should be the case for the other types of proof too.

It is clear therefore that students should practice the different proof methods to gain experience to become confident and importantly to enhance the capacity of their working memories. The only way in which students can engage successfully with proving then is to practice so that the proof structure can become part of their long term memory. Their working memory is then freed for other important functions of proving like looking for patterns in the given information. Furthermore neuro-science has shown the more you practice the more permanent connections are formed in the brain and thus long term memories are created.

The current South African NCS mathematics curriculum specifies that learners are to engage with proving activities in various content areas like number patterns and Euclidean geometry. The curriculum however does not make specific mention that students are to know what proof and proving mean and that there are different methods of proving and that each proof has a unique structure. The curriculum also does not prescribe that learners should be made aware of the type of proof they are employing in

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their proof activities. We however are of the opinion that learners should be taught which type of proof they are employing in all their proof activities. Furthermore learners should be made aware that each proof has a specific structure and that it is of utmost importance that they commit the structure of the proof to their long term memories. This is so that if with the passing of time they forget the specific proof examples they have dealt with they will still remember the structure of the proof and the methodology employed. More advanced levels of proving at school level should also include instruction on identifying conditions that will indicate that a specific proof cannot be applied.

8.7 Assessment and intervention strategies

The NCS prescribes four types of assessment namely baseline, diagnostic, formative and summative assessment. Baseline assessment is used to establish what skills and knowledge learners already have and is usually done at the start of a grade or learning cycle. Diagnostic assessment is used to uncover the cause or causes of a learning barrier and therefore assists educators to decide on support strategies. Alternatively it is used to discover what learning has not taken place so as to put intervention strategies in place. The goal of formative assessment is to provide feedback to the learner and to inform the teacher as to the progress of the learner. Summative assessment is used to judge the competence of learners and is therefore used for progression purposes.

One of the factors that contribute enormously to student deficiency in mathematics and especially in proof is the fact that although reasoning errors, misconceptions, gaps in prior knowledge, etc. are uncovered by assessment it is not addressed in the majority of cases. The following quote from Kutzler **[58]** provides a reason for this state of affairs: "The curriculum forces the teacher to continue with the next topic independent of the progress of individual students." What Kutzler is alluding to here is that teachers are forced by time constraints, volume of the content of the curriculum and other factors like class size to continue with the next topic although some learners might not have acquired the requisite competency in the previous topic. As indicated earlier what this then implies is that learners are passed on from grade to grade without acquiring the necessary competencies. This in turn will prevent students from building a complete understanding of mathematical concepts. The question is what can be done to alleviate these deficiencies?

How does one know if a student understood a mathematical lesson that was presented in class? In general student reasoning can be determined either by verbal responses or by written responses in answer to assessment. Hence teachers have to utilize either verbal questioning or paper and pencil assessment to determine students' levels of comprehension. Teacher questioning strategy whilst presenting lessons and also on completion of lessons is of utmost importance and should be based on the cognitive requirements of the topic under discussion. One of the aims of a questioning strategy obviously has to be to determine if the students understood the teaching, but also and most importantly to determine student misconceptions and gaps in prior knowledge. Once student errors in reasoning, misconceptions or gaps in prior knowledge are determined these have to be dealt with immediately so as to eliminate these problems, but also to

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prevent students progressing to the next grade with these misconceptions and incomplete understanding of mathematical concepts. So in essence remedial teaching should be an ongoing teaching strategy. It would therefore be in the interest of students and teachers to spend more time on baseline, diagnostic and formative assessment than on summative assessment. We are aware however that large class sizes and the administrative load of the average teacher in South Africa are inhibiting factors that prevent this from happening on an ongoing basis.

8.8 Mathematical language versus Everyday language

There is a definite difference between everyday language and mathematical language. Epp **[34]** argues about this as follows: " One reason students may have problems with formal mathematical reasoning is that certain forms of statements are open to different interpretations in informal and formal settings...By contrast, mathematical language is required to be unambiguous, with each grammatical construct having exactly one meaning." This difference between everyday and mathematical language complicates proof and proving even more since every conjecture and statement that has to be proved is given in precise mathematical language. Failure to interpret the statement correctly leads to all kinds of complications in proving. It is therefore incumbent upon proof instructors to teach students how to interpret mathematical statements. The results of our study have shown that instruction in the logic of compound and quantified statements provides an effective way to achieve this.

8.9 Pedagogical content knowledge

We have indicated in some of our earlier arguments that the main protagonists in the teaching and learning process are the teacher and the learner. Note we are not arguing that they are the only role players, but that they are the main ones. As a result of this it is imperative that the pedagogical content knowledge of teachers be improved. Pedagogical content knowledge refers to the knowledge that is required by teachers to know how to teach a specific concept in mathematics. Research done by both provincial and national education departments in South Africa indicate that the mathematical results need a lot of improvement. One of the reasons for the bad results in our opinion is that teachers currently in the system are not equipped pedagogically for the demands of the new curriculum both at the primary and high school level. This is corroborated by the TIMSS **[47]** report that showed that South African teachers are among the lowest qualified mathematics teachers of the 50 countries that participated in the study. What we are advocating is that much more attention should be given to those pedagogical skills that are required by the new curriculum. This cannot be accomplished by short once off workshops as is the case currently. Teachers in the system should be allowed study leave so that they can for extended periods study and research the pedagogical content knowledge that is required by the new curriculum. Alternatively in service training should be provided on an ongoing basis and not on a once off basis as is currently the case.

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8.10 Conclusion

It is clear that high school learners leave the school system with less than the expected competencies in terms of proof and proving. There can be no doubt that proof and proving is essential in mathematics. It has been shown that the cognitive abilities developed by reasoning in proof are abilities that can be employed in other domains of mathematics and also in other subject areas. Some of our earlier arguments have also indicated that the development of good science and mathematics teachers is crucial to the economic development of countries. All of these arguments therefore compel learning institutions to spend more time and effort to develop the necessary competencies in students in terms of proof and proving. The arguments in this paper therefore have to be seen as an attempt to provide some solutions and debate around the essential topic of proof and proving. Finally I would like to agree with Stylianides and Stylianides **[80]** that findings of different disciplines need to be harnessed to truly understand the learning process as far as proof and proving are concerned. We therefore suggest that in order to truly understand how learners reason in proof specifically and mathematics in general and hence to correct reasoning errors and to teach better one has to cross the divide between mathematics education, educational psychology and neuroscience.

8.11 Future Research

Based on the above arguments I am of the opinion that an area of research that requires our attention is the reasoning involved in the different kinds of activities in the mathematics classroom. Our suggestion is that the different kinds of reasoning should be investigated in the light of current findings of neuroscience, educational psychology and teaching methodologies so as to deliver a better pedagogical product, to improve the learning process in mathematics to ultimately make mathematics more understandable and enjoyable for both teacher and learner.

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APPENDICES

Appendix A1

UNIVERSITY OF THE WESTERN CAPE
DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

COURSE: APPLIED MATHEMATICS 112

NAME & SURNAME:…………………………………………………………….

STUDENT NO:……………………………. DATE:………………………

FIRST LANGUAGE:…………………………….

Read through the following puzzle and then try and solve it. Please do the puzzle on your own and we will allow time at a later stage to discuss your solution with your fellow students. Write your solution in the space provided. Explain in your own words the reasoning that you used to arrive at your solution.

PUZZLE I

In the back of an old cupboard you discover a note signed by a pirate famous for his bizarre sense of humour and love of logical puzzles. In the note he wrote that he had hidden treasure somewhere on the property. He listed five true statements (a – e below) and challenged the reader to use them to figure out the location of the treasure.

a) If this house is next to a lake, then the treasure is not in the kitchen.
b) If the tree in the front yard is an elm, then the treasure is in the kitchen.
c) This house is next to a lake.
d) The tree in the front yard is an elm or the treasure is buried under the flagpole.
e) If the tree in the backyard is an oak, then the treasure is in the garage.

Where is the treasure hidden?

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Appendix A2

UNIVERSITY OF THE WESTERN CAPE
DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

COURSE: APPLIED MATHEMATICS 112

NAME & SURNAME:……………………………………………………………….

STUDENT NO:……………………………. DATE:………………………

FIRST LANGUAGE:…………………………….

Read through the following puzzle and then try and solve it. Please do the puzzle on your own and we will allow time at a later stage to discuss your solution with your fellow students. Write your solution in the space provided. Explain in your own words the reasoning that you used to arrive at your solution.

PUZZLE III (POST TEST)

Your grandfather that is known for his sense of humour and love of logical puzzles left you a note. In the note he wrote that he had hidden your birthday present somewhere on one of his properties. He listed five true statements (a – e below) and challenged you to figure out the location of the present.

a) If this house is next to a main road, then the present is not in the attic.
b) If there is a swing in the yard then the present is in the study.
c) The yard has a lawn or the present is in the cupboard next to the stove.
d) If the yard has a lawn, then the present is in the attic.
e) The house is next to a main road.

Where is the present hidden?

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Appendix B1

UNIVERSITY OF THE WESTERN CAPE
DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

COURSE: APPLIED MATHEMATICS 112

NAME & SURNAME:……………………………………………………………….

STUDENT NO:……………………………. DATE:………………………

FIRST LANGUAGE:…………………………….

Read through the following puzzle and then try and solve it. Please do the puzzle on your own and we will allow time at a later stage to discuss your solution with your fellow students. Write your solution in the space provided. Explain in your own words the reasoning that you used to arrive at your solution.

KNIGHTS AND KNAVES (pre-test)

The logician Raymond Smullyan describes an island containing two types of people: knights who always tell the truth and knaves who always lie. You visit the island and are approached by two natives who speak to you as follows:

A says: B is a knight
B says: A and I are of opposite type.

What are A and B?

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Appendix B2

UNIVERSITY OF THE WESTERN CAPE
DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

COURSE: APPLIED MATHEMATICS 112

NAME & SURNAME:…………………………………………………………….

STUDENT NO:……………………………. DATE:………………………

FIRST LANGUAGE:…………………………….

Read through the following puzzle and then try and solve it. Please do the puzzle on your own and we will allow time at a later stage to discuss your solution with your fellow students. Write your solution in the space provided. Explain in your own words the reasoning that you used to arrive at your solution.

KNIGHTS AND KNAVES (post - test)

The logician Raymond Smullyan describes an island containing two types of people: knights who always tell the truth and knaves who always lie. You visit the island and are approached by two natives C and D but only C speaks.

C says: Both of us are knaves.

What are C and D?

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Appendix C1

Arguments with quantified statements (pre-test)

Rewrite the following statements in formal language

29.

1. No birds, except ostriches, are nine feet high.
2. There are no birds in this aviary that belong to anyone but me.
3. No ostrich lives on mince pies
4. I have no birds less than nine feet high.

30.

1. All writers who understand human nature are clever.
2. No one is a true poet unless he can stir the hearts of men.
3. Shakespeare wrote Hamlet.
4. No writer who does not understand human nature can stir the hearts of men.
5. None but a true poet could have written Hamlet.

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Appendix C2

UNIVERSITY OF THE WESTERN CAPE
DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS
APPLIED MATHEMATICS 112
CLASSTEST 6
DATE: 5 APRIL 2007

Arguments with quantified statements (post-test)

Rewrite the following statements in formal language

1 No bank closes before 3:30 unless it is a small bank.
2. No shark eats plankton unless it is a whale shark.
3. Students never study unless they have to prepare for a test.
4. None but a true gentleman will offer his seat to a lady on a bus.
5. None but a brave soldier will fight in a war

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Appendix D1

UNIVERSITY OF THE WESTERN CAPE
DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

COURSE: APPLIED MATHEMATICS 112

NAME & SURNAME:………………………………………………………….

STUDENT NO:…………………………….   DATE:………………………

FIRST LANGUAGE:…………………………….

Reorder the premises in the following argument to make it clear that the conclusion follows logically. It may be helpful to rewrite some of the statements in if – then form and to replace some statements by their contrapositives.

ARGUMENTS WITH QUANTIFIED STATEMENTS

PUZZLE I (pre – test)

11. When I work a logic example without grumbling, you may be sure it is one I understand.
12. The arguments in these examples are not arranged in regular order like the ones I am used to.
13. No easy examples make my head ache.
14. I can't understand examples if the arguments are not arranged in regular order like the ones I am used to.
15. I never grumble at an example unless it gives me a headache.

$\therefore$ These examples are not easy.

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Appendix D2

UNIVERSITY OF THE WESTERN CAPE
DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

COURSE: APPLIED MATHEMATICS 112

NAME & SURNAME:……………………………………………………………….

STUDENT NO:……………………………. DATE:………………………

FIRST LANGUAGE:…………………………….

Reorder the premises in the following argument to make it clear that the conclusion follows logically. It may be helpful to rewrite some of the statements in *if – then* form and to replace some statements by their *contrapositives.*

ARGUMENTS WITH QUANTIFIED STATEMENTS

PUZZLE II (post – test)

16. There is no box of mine here that I dare open.
17. My writing-desk is made of rose-wood.
18. All my boxes are painted, except what are here.
19. There is no box of mine that I dare not open, unless it is full of live scorpions.
20. All my rose-wood boxes are unpainted.

$\therefore$ My writing-desk is full of live scorpions.

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Appendix E1

UNIVERSITY OF THE WESTERN CAPE
DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

COURSE: APPLIED MATHEMATICS 112

NAME & SURNAME:…………………………………………………………….

STUDENT NO:……………………………. DATE:………………………

FIRST LANGUAGE:…………………………….

SET THEORY
(pre-test)

For all sets *A, B* and *C* prove the following:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

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Appendix E2

UNIVERSITY OF THE WESTERN CAPE
DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

COURSE: APPLIED MATHEMATICS 112

NAME & SURNAME:………………………………………………………….

STUDENT NO:………………………….   DATE:………………………

FIRST LANGUAGE:…………………………..

SET THEORY
(post-test)

For all sets *A*, *B* and *C* prove the following:

$(A \cup B) \cup C = A \cup (B \cup C)$

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Appendix F1

UNIVERSITY OF THE WESTERN CAPE
DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

COURSE: APPLIED MATHEMATICS 112

NAME & SURNAME:…………………………………………………………….

STUDENT NO:………………………….   DATE:………………………

FIRST LANGUAGE:…………………………….

METHOD OF DIRECT PROOF AND DIVISIBILITY
(pre-test)

Prove the following by using the method of direct proof:

For all integers a, b, and c, if a / b and a / c then $a / (b+c)$

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Appendix F2

UNIVERSITY OF THE WESTERN CAPE
DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

COURSE: APPLIED MATHEMATICS 112

NAME & SURNAME:……………………………………………………………….

STUDENT NO:…………………………….   DATE:………………………

FIRST LANGUAGE:…………………………….

METHOD OF DIRECT PROOF AND DIVISIBILITY
(post-test)

Prove the following by using the method of direct proof:
For all integers *a, b*, and *c,* if a / b and a / c then $a / (b - c)$

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Appendix G1

UNIVERSITY OF THE WESTERN CAPE
DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

COURSE: APPLIED MATHEMATICS 112

NAME & SURNAME:...

STUDENT NO:................................. DATE:...........................

FIRST LANGUAGE:.................................

METHOD OF DIRECT PROOF I
(pre-test)

Prove the following statements using the method of direct proof:

(i) For all integers *m*, if $m \rangle 1, then \quad 0 \langle \frac{1}{m} \langle 1$

(ii) The difference of any two rational numbers is a rational number.

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Appendix G2

UNIVERSITY OF THE WESTERN CAPE
DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

COURSE: APPLIED MATHEMATICS 112

NAME & SURNAME:……………………………………………………………….

STUDENT NO:…………………………….   DATE:………………………

FIRST LANGUAGE:…………………………….

METHOD OF DIRECT PROOF I
(post-test)

Prove the following statements using the method of direct proof:

(i) For all real numbers x, if $0 \langle x \langle 1,$ then $x^2 \langle x$

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Appendix H1

UNIVERSITY OF THE WESTERN CAPE
DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

COURSE: APPLIED MATHEMATICS 112

NAME & SURNAME:………………………………………………………………….

STUDENT NO:……………………………….   DATE:………………………

FIRST LANGUAGE:……………………………….

METHOD OF INDUCTION
(Pre-test)

Use mathematical induction to prove that:
$2 + 4 + 6 + \ldots 2n = n^2 + n,$
for all integers $n \geq 1$

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Appendix H2

UNIVERSITY OF THE WESTERN CAPE
DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

COURSE: APPLIED MATHEMATICS 112

NAME & SURNAME:……………………………………………………………….

STUDENT NO:……………………………. DATE:………………………

FIRST LANGUAGE:…………………………….

METHOD OF INDUCTION
(Post-test)

Use mathematical induction to prove that:
$1 + 5 + 9 + ...(4n-3) \ = \ n(2n-1),$
for all integers $n \geq 1$

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Appendix I1

UNIVERSITY OF THE WESTERN CAPE DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

COURSE: APPLIED MATHEMATICS 112

NAME & SURNAME:……………………………………………………………….

STUDENT NO:……………………………. DATE:………………………

FIRST LANGUAGE:…………………………….

STRONG MATHEMATICAL INDUCTION
(RECURSIVE SEQUENCES)
(PRE-TEST)

1 Suppose that c_0 , c_1 , c_2 ,... is a sequence defined as follows:

$c_0 = 2, \quad c_1 = 4, \quad c_2 = 6,$

$c_k = 5c_{k-3}$ *for all integers* $k \geq 3$

Prove that c_n is even for all integers $n \geq 0$

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Appendix I2

UNIVERSITY OF THE WESTERN CAPE
DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

COURSE: APPLIED MATHEMATICS 112

NAME & SURNAME:……………………………………………………………….

STUDENT NO:……………………………. DATE:………………………

FIRST LANGUAGE:…………………………….

STRONG MATHEMATICAL INDUCTION
(RECURSIVE SEQUENCES)
(Post-test)

1 Suppose $a_1, a_2, a_3, \ldots$ is a sequence defined as follows:

$a_1 = 1, \quad a_2 = 3,$

$a_k = a_{k-2} + 2a_{k-1} \quad \textit{for all integers } k \geq 3.$

Prove that a_n is odd for all integers $n \geq 1$

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Appendix J

QUESTIONAIRRE FOR NCS TEACHERS
2007

Degree/ Diploma and state where it was obtained:……………………………………

Highest qualification in Mathematics:………………………………………………

School :……………………………………………………………………………

1. What teaching approach (methodology) do you use when teaching mathematical proofs?

   ……………………………………………………………………………………

2. Are you aware of other teaching approaches besides the one you are using? (yes or no). If you are aware of other approaches, name them.

   ……………………………………………………………………………………

3. What type of proof is used in the grade 10 to 12 mathematics curriculum?

   ……………………………………………………………………………………

4. Do you know of other types of proof besides the proof that is used in the school curriculum? If you know other types, name them.

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5. Have you studied mathematical logic or do you know about mathematical logic? Please specify.

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6. If you did study mathematical logic or if you know about it do you know where in school proofs logic is used? If you do know, name at least one example.

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Appendix K

PARTICIPANT CONSENT FORM

VERIFICATION OF ADULT INFORMED CONSENT FOR OWN PARTICIPATION

I,……………………………………………………………………………………………….
(Please print full name and surname)

voluntarily give my consent to serve as a participant in the study entitled:

PROVING MATHEMATICAL STATEMENTS: BY FIRST YEAR STUDENTS AT THE UNIVERSITY OF THE WESTERN CAPE.

I have received a satisfactory explanation of the general purpose and process of this study, as well as a description of what I will be asked to do and the conditions that I will be exposed to.

It is my understanding that my participation in this study is voluntary and I will receive no remuneration for my participation.

It is further my understanding that I may terminate my participation in this study at any time and that any data obtained will be held confidential. I am aware that the researcher has to report to his supervisor and that all data collected will be accessible to the supervisor as well.

Signature of participant:……………………………………………………………………….

Date:………………………………………………………………………………………..

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Appendix L

Navrae
Enquiries **Dr RS Cornelissen**
IMibuzo

Telefoon
Telephone **(021) 467-2286**
IFoni

Faks
Fax **(021) 425-7445**
IFeksi

Verwysing
Reference **20070420-0041**
ISalathiso

Wes-Kaap Onderwysdepartement

Western Cape Education Department

ISebe leMfundo leNtshona Koloni

Mr Bruce May
21 Padrone Crescent
Wavecrest
STRANDFONTEIN
7785

Dear Mr B. May

RESEARCH PROPOSAL: PROVING MATHEMATICAL STATEMENTS: BY FIRST YEAR STUDENTS AT THE UNIVERSITY OF THE WESTERN CAPE.

Your application to conduct the above-mentioned research in schools in the Western Cape has been approved subject to the following conditions:

1. Principals, educators and learners are under no obligation to assist you in your investigation.
2. Principals, educators, learners and schools should not be identifiable in any way from the results of the investigation.
3. You make all the arrangements concerning your investigation.
4. Educators' programmes are not to be interrupted.
5. The Study is to be conducted from **23rd April 2007 to 22nd June 2007**.
6. No research can be conducted during the fourth term as schools are preparing and finalizing syllabi for examinations (October to December 2007).
7. Should you wish to extend the period of your survey, please contact Dr R. Cornelissen at the contact numbers above quoting the reference number.
8. A photocopy of this letter is submitted to the Principal where the intended research is to be conducted.
9. Your research will be limited to the list of schools as submitted to the Western Cape Education Department.
10. A brief summary of the content, findings and recommendations is provided to the Director: Education Research.
11. The Department receives a copy of the completed report/dissertation/thesis addressed to:
    The Director: Education Research
    Western Cape Education Department
    Private Bag X9114
    CAPE TOWN
    8000

We wish you success in your research.

Kind regards.

Signed: Ronald S. Cornelissen
for: **HEAD: EDUCATION**
DATE: 23rd April 2007

MELD ASSEBLIEF VERWYSINGSNOMMERS IN ALLE KORRESPONDENSIE / PLEASE QUOTE REFERENCE NUMBERS IN ALL CORRESPONDENCE / NCEDA UBHALE IINOMBOLO ZESALATHISO KUYO YONKE IMBALELWANO

GRAND CENTRAL TOWERS, LAER-PARLEMENTSTRAAT, PRIVAATSAK X9114, KAAPSTAD 8000
GRAND CENTRAL TOWERS, LOWER PARLIAMENT STREET, PRIVATE BAG X9114, CAPE TOWN 8000

WEB: http://wced.wcape.gov.za

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